STABILITY OF BOOLEAN FUNCTION CLASSES WITH RESPECT TO CLONES OF LINEAR FUNCTIONS

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This paper is dedicated to Maurice Pouzet to whom we are deeply thankful for his guidance, friendship, knowledgeable support, and for being always a source of great motivation and inspiration.

Abstract. We consider classes of Boolean functions stable under compositions both from the right and from the left with clones. Motivated by the question how many properties of Boolean functions can be defined by means of linear equations, we focus on stability under compositions with the clone of linear idempotent functions. It follows from a result by Sparks that there are countably many such linearly definable classes of Boolean functions. In this paper, we refine this result by completely describing these classes. This work is tightly related with the theory of function minors, stable classes, clonoids, and hereditary classes, topics that have been widely investigated in recent years by several authors including Maurice Pouzet and his coauthors.

1. Introduction

This paper is a study of classes of functions of several arguments from a set A to a set B that are closed under composition from the right with a clone C_1 on A and under composition from the left with a clone C_2 on B, in brief, (C_1, C_2)-*stable* classes of functions. Special instances of the notion of (C_1, C_2)-stability appear in the literature. For example, if both C_1 and C_2 are clones of projections on the respective sets, then we get *minor-closed classes* or *minions* or *equational classes* (see Pippenger [14], Ekin et al. [8]). If C_1 the clone of projections and C_2 is the clone of an algebra $\mathbf{B}$, then we get *clonoids* with source set A and target algebra $\mathbf{B}$ (see Aichinger and Mayr [1]).

If both C_1 and C_2 are equal to the clone L_c of idempotent linear functions on $\{0, 1\}$, then the (C_1, C_2)-stable classes are exactly the classes of Boolean functions definable by linear equations (see [4]). It was already observed in [4] that there are infinitely many such linearly definable classes, but it remained an open question whether there are countably or uncountably many such classes and exactly what these classes are.

More generally, we would like to describe (C_1, C_2)-stable classes. This problem seems unfeasible in full generality, since there are uncountably many clones on sets with at least three elements (see Yanov and Muchnik [20]). This fact led us to considering (C_1, C_2)-stability for clones C_1 and C_2 on the two-element set. Motivated by linear definability, we focus on clones containing the clone L_c.

We show that that there are a countably infinite number of $(\mathsf{L}_\mathsf{c}, \mathsf{L}_\mathsf{c})$-stable classes (in brief, L_c-*stable* classes), and we provide an explicit description thereof. More precisely, the paper is organized as follows.

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- Section 2: We provide the basic definitions and preliminary results that are needed in the sequel.
- Section 3: We establish some auxiliary tools for studying (C_1, C_2)-stability.
- Section 4: We make a little diversion to clones on arbitrary finite fields, and we describe the L-stable classes, where L denotes the clone of all linear functions on $\mathbb{F}_q$.
- Section 5: We define various properties of Boolean functions that are needed for describing the L_c-stable classes.
- Section 6: We present our main result: an explicit description of the L_c-stable classes of Boolean functions. The proof has two parts. First we show that the listed classes are L_c-stable; this is straightforward verification. The more difficult part of the proof is to show that there are no further L_c-stable classes.
- Section 7: With the help of the result on L_c-stable classes, we obtain with little effort also a description of (C_1, C_2)-stable classes for clones C_1 and C_2, where C_1 is arbitrary and $\mathsf{L}_\mathrm{c} \subseteq C_2$
- Section 8: We make some concluding remarks and indicate directions for future research.

The main results of this paper were presented without proofs in the 1st International Conference on Algebras, Graphs and Ordered Sets (ALGOS 2020) [7]. The reader should be cautious about the fact that some notation and terminology have been slightly changed from the conference paper.

2. Preliminaries

The symbols $\mathbb{N}$ and $\mathbb{N}_+$ denote the set of all nonnegative integers and the set of all positive integers, respectively. For any $n \in \mathbb{N}$, the symbol $[n]$ denotes the set $\{\, i \in \mathbb{N} \mid 1 \leq i \leq n \,\}$.

Definition 2.1. Let A and B be sets. A mapping of the form $f \colon A^n \to B$ for some $n \in \mathbb{N}_+$ is called a *function of several arguments from A to B* (or simply a *function*). The number n is called the *arity* of f and denoted by $\mathrm{ar}(f)$. If $A = B$, then such a function is called an *operation on A*. We denote by $\mathcal{F}_{AB}$ and $\mathcal{O}_A$ the set of all functions of several arguments from A to B and the set of all operations on A, respectively. For any $n \in \mathbb{N}_+$, we denote by $\mathcal{F}_{AB}^{(n)}$ the set of all n-ary functions in $\mathcal{F}_{AB}$, and for any $C \subseteq \mathcal{F}_{AB}$, we let $C^{(n)} := C \cap \mathcal{F}_{AB}^{(n)}$ and call it the *n-ary part* of C.

Definition 2.2. For $b \in B$ and $n \in \mathbb{N}$, the n-ary *constant function* $\mathrm{c}_b^{(n)} \colon A^n \to B$ is given by the rule $(a_1, \dots, a_n) \mapsto b$ for all $a_1, \dots, a_n \in A$. In the case when $A = B$, for $n \in \mathbb{N}$ and $i \in [n]$, the i-th n-ary *projection* $\mathrm{pr}_i^{(n)} \colon A^n \to A$ is given by the rule $(a_1, \dots, a_n) \mapsto a_i$ for all $a_1, \dots, a_n \in A$.

Definition 2.3. Let $f \colon A^n \to B$ and $i \in [n]$. The i-th argument is *essential* in f if there exist $a_1, \dots, a_n, a'_i \in A$ such that

$$f(a_1, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n).$$

An argument that is not essential is *fictitious.* The *essential arity* of f is the number of its essential arguments.

Definition 2.4. Let $f \colon B^n \to C$ and $g_1, \dots, g_n \colon A^m \to B$. The *composition* of f with $g_1, \dots, g_n$ is the function $f(g_1, \dots, g_n) \colon A^m \to C$ given by the rule

$$f(g_1, \dots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a}))$$

for all $\mathbf{a} \in A^m$. The function f is called the *outer function* and $g_1, \dots, g_n$ are called the *inner functions* of the composition.

Definition 2.5. Let $f\colon A^n \to B$ and $\sigma\colon [n] \to [m]$. Define the function $f_\sigma\colon A^m \to B$ by the rule

$$f_\sigma(a_1, \dots, a_m) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)}),$$

for all $a_1, \dots, a_m \in A$. Such a function f_σ is called a *minor* of f, formed via the *minor formation map* σ. Intuitively, minors of f are all those functions that can be obtained from f by manipulation of its arguments: permutation of arguments, introduction of fictitious arguments, identification of arguments. It is clear from the definition that the minor f_σ can be obtained as a composition of f with m-ary projections on A:

$$f_\sigma = f(\mathrm{pr}^{(m)}_{\sigma(1)}, \dots, \mathrm{pr}^{(m)}_{\sigma(n)}).$$

An important special case of minors is the identification of a pair of arguments. This is obtained with minor formation maps of the following form: for $i, j \in [n]$ with $i < j$, let $\sigma_{ij}\colon [n] \to [n-1]$ be given by

$$\sigma_{ij}(m) = \begin{cases} m, & \text{if } m < j, \\ i, & \text{if } m = j, \\ m-1, & \text{if } m > j. \end{cases}$$

We call such a map σ_{ij} an *identification map,* and we write f_{ij} for $f_{\sigma_{ij}}$. More explicitly,

$$f_{ij}(a_1, \dots, a_{n-1}) = f(a_1, \dots, a_i, \dots, a_{j-1}, a_i, a_j, \dots, a_{n-1}).$$

We write $f \leq g$ if f is a minor of g. The minor relation $\leq$ is a quasiorder (a reflexive and transitive relation) on $\mathcal{F}_{AB}$, and it induces an equivalence relation $\equiv$ on $\mathcal{F}_{AB}$ and a partial order on the quotient $\mathcal{F}_{AB}/{\equiv}$ in the usual way: $f \equiv g$ if $f \leq g$ and $g \leq f$, and $f/{\equiv} \leq g/{\equiv}$ if $f \leq g$.

The effect of successive formations of minors is captured by the composition of minor-forming maps.

Lemma 2.6. *Let $f\colon A^n \to B$, $\sigma\colon [n] \to [m]$, and $\tau\colon [m] \to [\ell]$. Then $(f_\sigma)_\tau = f_{\tau \circ \sigma}$.*

Proof. For all $a_1, \dots, a_\ell \in A$, we have

$$\begin{aligned}(f_\sigma)_\tau(a_1, \dots, a_\ell) &= (f_\sigma)(a_{\tau(1)}, \dots, a_{\tau(m)}) = f(a_{\tau(\sigma(1))}, \dots, a_{\tau(\sigma(n))}) \\ &= f(a_{(\tau\circ\sigma)(1)}, \dots, a_{(\tau\circ\sigma)(n)}) = f_{\tau\circ\sigma}(a_1, \dots, a_\ell). \qquad \square\end{aligned}$$

Remark 2.7. It is well known that any function can be decomposed into a surjection and an injection. This obviously holds for minor formation maps $\sigma\colon [n] \to [m]$; we obtain $\sigma = \rho \circ \tau$ where $\tau\colon [n] \to [\ell]$ is surjective and $\rho\colon [\ell] \to [m]$ is injective. Moreover, as explained in [12, Section 2.2], we can choose the surjective map τ so that it is a composition of a number of identification maps: $\tau = \sigma_{i_k j_k} \circ \dots \circ \sigma_{i_1 j_1}$ (we regard the empty composition as the identity map on $[\ell]$).

Intuitively, this means that any minor of a function $f\colon A^n \to B$ can be formed by first successively identifying pairs of arguments, and then introducing fictitious arguments and permuting arguments.

Composition of functions satisfies the so-called superassociative law. Consequently, formation of minors commutes with composition.

Lemma 2.8. *Let $f\colon C^n \to D$, $g_1, \dots, g_n\colon B^m \to C$, $h_1, \dots, h_m \in A^\ell \to B$. Then $(f(g_1, \dots, g_n))(h_1, \dots, h_m) = f(g_1(h_1, \dots, h_m), \dots, g_n(h_1, \dots, h_m))$. Consequently, for any $\sigma\colon [\ell] \to [m]$, we have $(f(g_1, \dots, g_n))_\sigma = f((g_1)_\sigma, \dots, (g_n)_\sigma))$.*

Proof. For any $\mathbf{a} \in A^\ell$, we have

$$\begin{aligned}
&(f(g_1, \dots, g_n))(h_1, \dots, h_m)(\mathbf{a}) = (f(g_1, \dots, g_n))(h_1(\mathbf{a}), \dots, h_m(\mathbf{a})) \\
&= f(g_1(h_1(\mathbf{a}), \dots, h_m(\mathbf{a})), \dots, g_n(h_1(\mathbf{a}), \dots, h_m(\mathbf{a}))) \\
&= f(g_1(h_1, \dots, h_m)(\mathbf{a}), \dots, g_n(h_1, \dots, h_m)(\mathbf{a})) \\
&= f(g_1(h_1, \dots, h_m), \dots, g_n(h_1, \dots, h_m))(\mathbf{a}).
\end{aligned}$$

The statement about minors follows by taking $h_i := \mathrm{pr}^{(\ell)}_{\sigma(i)}$, $1 \leq i \leq m$. □

The notion of functional composition extends naturally to classes of functions.

Definition 2.9. Let $C \subseteq \mathcal{F}_{BC}$ and $K \subseteq \mathcal{F}_{AB}$. The *composition* of C with K is defined as

$$CK := \{\, f(g_1, \dots, g_n) \mid f \in C^{(n)},\, g_1, \dots, g_n \in K^{(m)},\, n, m \in \mathbb{N}_+ \,\}.$$

Remark 2.10. It follows immediately from the definition of function class composition that if $C, C' \subseteq \mathcal{F}_{BC}$ and $K, K' \subseteq \mathcal{F}_{AB}$ satisfy $C \subseteq C'$ and $K \subseteq K'$, then $CK \subseteq C'K'$.

Lemma 2.11. *For any $C_1, C_2 \subseteq \mathcal{F}_{BC}$, $K \subseteq \mathcal{F}_{AB}$, it holds that $(C_1 \cap C_2)K \subseteq C_1K \cap C_2K$ and $(C_1 \cup C_2)K = C_1K \cup C_2K$.*

Proof. We clearly have $(C_1 \cap C_2)K = (C_1 \cap C_2)K \cap (C_1 \cap C_2)K \subseteq C_1K \cap C_2K$ and $C_1K \cup C_2K \subseteq (C_1 \cup C_2)K \cup (C_1 \cup C_2)K = (C_1 \cup C_2)K$. In order to prove the inclusion $(C_1 \cup C_2)K \subseteq C_1K \cup C_2K$, let $h \in (C_1 \cup C_2)K$. Then $h = f(g_1, \dots, g_n)$ for some $f \in C_1 \cup C_2$ and $g_1, \dots, g_2 \in K$. Since $f \in C_1$ or $f \in C_2$, we have that $f(g_1, \dots, g_n)$ belongs to C_1K or C_2K; therefore $h = f(g_1, \dots, g_n) \in C_1K \cup C_2K$. □

Remark 2.12. The inclusion $C_1K \cap C_2K \subseteq (C_1 \cap C_2)K$ does not hold in general. For a counterexample, let $C_1 := \{\pi^{(1)}_1\}$, $C_2 := \{c^{(1)}_0\}$, $K := \{c^{(1)}_0\}$, subsets of $\mathcal{O}_{\{0,1\}}$, where $c^{(1)}_0$ denotes the unary constant function taking value 0. Then $C_1K = C_2K = \{c^{(1)}_0\}$, so $C_1K \cap C_2K = \{c^{(1)}_0\}$, but $(C_1 \cap C_2)K = \emptyset$ because $C_1 \cap C_2 = \emptyset$.

Definition 2.13. A class $C \subseteq \mathcal{O}_A$ is called a *clone* on A if $CC \subseteq C$ and C contains all projections. The set of all clones on A is a closure system in which the greatest and least elements are the clone $\mathcal{O}_A$ of all operations on A and the clone of all projections on A, respectively. For any $K \subseteq \mathcal{O}_A$, we denote by $\langle K \rangle$ the clone generated by K, i.e., the smallest clone on A containing K.

Definition 2.14. Let $K \subseteq \mathcal{F}_{AB}$, $C_1 \subseteq \mathcal{O}_A$, and $C_2 \subseteq \mathcal{O}_B$. We say that K is *stable under right composition* with C_1 if $KC_1 \subseteq K$, and that K is *stable under left composition* with C_2 is $C_2K \subseteq K$. If both $KC_1 \subseteq K$ and $C_2K \subseteq K$ hold, we say that K is (C_1, C_2)-*stable.* If $K, C \subseteq \mathcal{O}_A$ and K is (C, C)-stable, we say that K is C-*stable.*

The set of all (C_1, C_2)-stable subsets of $\mathcal{F}_{AB}$ constitutes a closure system, and for any $K \subseteq \mathcal{F}_{AB}$, we denote by $\langle K \rangle_{(C_1,C_2)}$ the (C_1, C_2)-*closure* of K, i.e., the smallest (C_1, C_2)-stable class containing K. We also write $\langle K \rangle_C$ for $\langle K \rangle_{(C,C)}$ and call it the C-*closure* of K.

Remark 2.15. A set $K \subseteq \mathcal{F}_{AB}$ is minor-closed if and only if it is stable under right composition with the set of all projections on A. Every clone is minor-closed. A clone C is (C, C)-stable.

Lemma 2.16. *Let C_1 and C'_1 be clones on A and C_2 and C'_2 clones on B such that $C_1 \subseteq C'_1$ and $C_2 \subseteq C'_2$. Then for every $K \subseteq \mathcal{F}_{AB}$, it holds that if K is (C'_1, C'_2)-stable then K is (C_1, C_2)-stable.*

Proof. Assume that K is (C'_1, C'_2)-stable. Then, in view of Remark 2.10, we have $KC_1 \subseteq KC'_1 \subseteq K$ and $C_2K \subseteq C'_2K \subseteq K$, i.e., K is (C_1, C_2)-stable. □

3. Stability and generators

The task of verifying whether a function class is stable under right or left compositions with certain clones may appear complicated because the defining conditions involve compositions with arbitrary members of each clone. We now develop helpful tools that simplify this task.

For right stability, it is enough to consider closure under minors and certain simple compositions involving only generators of the clone. In order to formalize this, let us consider the elementary superposition operations ζ (cyclic shift of arguments), τ (transposition of the first two arguments), Δ (identification of arguments or diagonalization), ∇ (introduction of a fictitious argument or cylindrification), and $*$ (composition) defined by Mal'cev [13] (see also [11, Section II.1.2]). The algebra $(\mathcal{O}_A; \zeta, \tau, \Delta, \nabla, *)$ is called the *iterative function algebra* on A, and its subuniverses are called *closed classes.* Closed classes containing all projections are precisely the clones on A.

Let $F \subseteq \mathcal{O}_A$ and $f \in \mathcal{O}_A$. We say that f is a *superposition* of F if f can be obtained from the members F by a finite number of applications of the operations ζ, τ, Δ, ∇, $*$.

Lemma 3.1. *For any $f \in \mathcal{O}_A^{(n)}$ and $g_1, \dots, g_n \in \mathcal{O}_A^{(m)}$, the composition $f(g_1, \dots, g_n)$ is a superposition of $\{f, g_1, \dots, g_n\}$.*

Proof. Let $f_0 := (\zeta f) * g_n$, and For $i = 1, \dots, n-1$, let $f_i := (\zeta f_{i-1}) * g_{n-i}$. Then

$$\begin{aligned}
&f_1(x_1, \dots, x_{n+m-1}) \\
&= (\zeta f)(g_n(x_1, \dots, x_m), x_{m+1}, \dots, x_{n+m-1}) \\
&= f(x_{m+1}, \dots, x_{n+m-1}, g_n(x_1, \dots, x_m)), \\
&f_2(x_1, \dots, x_{n+2m-2}) \\
&= (\zeta f_1)(g_{n-1}(x_1, \dots, x_m), x_{m+1}, \dots, x_{n+2m-2}) \\
&= f_1(x_{m+1}, \dots, x_{n+2m-2}, g_{n-1}(x_1, \dots, x_m)) \\
&= f(x_{2m+1}, \dots, x_{n+2m-2}, g_{n-1}(x_1, \dots, x_m), g_n(x_{m+1}, \dots, x_{2m})), \\
&f_3(x_1, \dots, x_{n+3m-3}) \\
&= (\zeta f_2)(g_{n-2}(x_1, \dots, x_m), x_{m+1}, \dots, x_{n+3m-3}) \\
&= f_2(x_{m+1}, \dots, x_{n+3m-3}, g_{n-2}(x_1, \dots, x_m)) \\
&= f(x_{3m+1}, \dots, x_{n+3m-3}, \\
&\qquad g_{n-2}(x_1, \dots, x_m), g_{n-1}(x_{m+1}, \dots, x_{2m}), g_n(x_{2m+1}, \dots, x_{3m})), \\
&\vdots \\
&f_n(x_1, \dots, x_{nm}) \\
&= f(g_1(x_1, \dots, x_m), g_2(x_{m+1}, \dots, x_{2m}), \dots, g_n(x_{(n-1)m+1}, \dots, x_{nm})).
\end{aligned}$$

Let θ be the composition of elementary operations that identifies arguments x_i and x_j if and only if $i \equiv j \pmod{m}$. Then

$$\begin{aligned}
\theta f_n(x_1, \dots, x_m) &= f(g_1(x_1, \dots, x_m), g_2(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)) \\
&= f(g_1, \dots, g_n)(x_1, \dots, x_m).
\end{aligned}$$

By construction, the functions $f_1, \dots, f_n$ and $\theta f_n = f(g_1, \dots, g_n)$ are superpositions of $\{f, g_1, \dots, g_n\}$. $\square$

Lemma 3.2. *Let $F \subseteq \mathcal{O}_A$. Let C be a clone on A, and let G be a generating set of C. Then the following conditions are equivalent.*

(i) $FC \subseteq F$

(ii) *F is minor-closed and $f * g \in F$ whenever $f \in F$ and $g \in C$.*
(iii) *F is minor-closed and $f * g \in F$ whenever $f \in F$ and $g \in G$.*

Proof. (i) $\implies$ (iii): For any $f \in F$, any minor of f is of the form $f(\mathrm{pr}_{i_1}^{(m)}, \dots, \mathrm{pr}_{i_m}^{(m)})$, for some $m \in \mathbb{N}$ and $i_1, \dots, i_m \in [m]$. Since all projections are members of the clone C, we have $f(\mathrm{pr}_{i_1}^{(m)}, \dots, \mathrm{pr}_{i_m}^{(m)}) \in FC \subseteq F$. Thus F is minor-closed.

Let $g \in G$ and define $g' := g(\mathrm{pr}_1^{(m+n-1)}, \dots, \mathrm{pr}_m^{(m+n-1)})$. Then $g' \in C$, and we have $f * g = f(g', \mathrm{pr}_{m+1}^{(m+n-1)}, \dots, \mathrm{pr}_{m+n-1}^{(m+n-1)}) \in FC \subseteq F$.

(iii) $\implies$ (ii): Let $g \in C$. If g is a projection, then for every $f \in F$, the function $f * g$ is a minor of f, obtained by introducing fictitious arguments, so $f * g \in F$ because F is minor-closed. If g is not a projection, then g is a superposition of G, that is, there is a term t, say ℓ-ary, in the language of iterative algebras and $h_1, \dots, h_\ell \in G$ such that $t^{\mathcal{O}_A}(h_1, \dots, h_\ell) = g$. We prove by induction on the structure of the term t that for every $f \in F$ it holds that $f * g \in F$. If $t = x_i$, then $t^{\mathcal{O}_A}(h_1, \dots, h_\ell) = h_i \in G$, and we have $f * h_i \in F$ by assumption. Consider then the case that $t = \varphi u$, where $\varphi \in \{\zeta, \tau, \Delta, \nabla\}$ and u is a term, and assume that $f * u^{\mathcal{O}_A}(h_1, \dots, h_\ell) \in F$ for every $f \in F$. Then also $f * t^{\mathcal{O}_A}(h_1, \dots, h_\ell) = f * \varphi u^{\mathcal{O}_A}(h_1, \dots, h_\ell) \in F$ for every $f \in F$, because F is minor-closed and the following identities hold for any functions f and h (say h is n-ary):

$$
\begin{aligned}
f * \zeta h &= \pi_{(1\ 2\ \cdots\ n)}(f * h), \\
f * \tau h &= \tau(f * h), \\
f * \Delta h &= \Delta(f * h), \\
f * \nabla h &= \nabla(f * h).
\end{aligned}
$$

Finally, consider the case that $t = u * v$, and assume that $f * u^{\mathcal{O}_A}(h_1, \dots, h_\ell) \in F$ and $f * v^{\mathcal{O}_A}(h_1, \dots, h_\ell) \in F$ for every $f \in F$. Then also $f * t^{\mathcal{O}_A}(h_1, \dots, h_\ell) = f * (u^{\mathcal{O}_A}(h_1, \dots, h_\ell) * v^{\mathcal{O}_A}(h_1, \dots, h_\ell)) = (f * u^{\mathcal{O}_A}(h_1, \dots, h_\ell)) * v^{\mathcal{O}_A}(h_1, \dots, h_\ell) \in F$ for every $f \in F$.

(ii) $\implies$ (i): Let $f \in F^{(n)}$ and $g_1, \dots, g_n \in C^{(m)}$. A simple inductive argument shows that, in the construction of $f(g_1, \dots, g_n)$ as a superposition of $\{f, g_1, \dots, g_n\}$ given in the proof of Lemma 3.1, the functions f_i are in F, because F is minor-closed and each f_i is of the form $\zeta\varphi * \gamma$ for some $\varphi \in F$ and $\gamma \in G$. Finally, $f(g_1, \dots, g_n) = \theta f_n \in F$, because F is minor-closed. $\square$

For left stability, it is enough to consider compositions with generators of the clone.

Lemma 3.3. *Let $F \subseteq \mathcal{O}_A$. Let C be a clone on A, and let G be a generating set of C. Then the following conditions are equivalent.*

(i) $CF \subseteq F$
(ii) *$g(f_1, \dots, f_n) \in F$ whenever $g \in C^{(n)}$ and $f_1, \dots, f_n \in F^{(m)}$ for some $n, m \in \mathbb{N}$.*
(iii) *$g(f_1, \dots, f_n) \in F$ whenever $g \in G^{(n)}$ and $f_1, \dots, f_n \in F^{(m)}$ for some $n, m \in \mathbb{N}$.*

Proof. (i) $\iff$ (ii): Holds by the definition of function class composition.

(ii) $\implies$ (iii): Obvious.

(iii) $\implies$ (ii): Let $g \in C$. Then there is a term t of the language of the algebra $\mathbf{A} = (A; G)$ such that $g = t^{\mathbf{A}}$. We prove the claim by induction on the structure of the term t. Let $f_1, \dots, f_n \in F^{(m)}$. The inductive basis holds, because if $t = x_i$, then $t^{\mathbf{A}} = \mathrm{pr}_i^{(n)}$, and we have $\mathrm{pr}_i^{(n)}(f_1, \dots, f_n) = f_i \in F$. Consider now the case when $t = h(t_1, \dots, t_\ell)$ for some $h \in G$ and terms $t_1, \dots, t_\ell$, and assume that for

$i \in \{1, \dots, \ell\}$, we have already shown that $t_i^{\mathbf{A}}(f_1, \dots, f_n) \in F$. It then follows from superassociativity and our assumptions that

$$\begin{aligned} t^{\mathbf{A}}(f_1, \dots, f_n) &= h^{\mathbf{A}}(t_1^{\mathbf{A}}, \dots, t_\ell^{\mathbf{A}})(f_1, \dots, f_n) \\ &= h^{\mathbf{A}}(t_1^{\mathbf{A}}(f_1, \dots, f_n), \dots, t_\ell^{\mathbf{A}}(f_1, \dots, f_n)) \in F. \end{aligned}$$ □

Let us record here a simple yet useful observation on the C-stable class generated by a projection.

Lemma 3.4. *For any clone C, $\langle \mathrm{pr}_1^{(1)} \rangle_C = C$.*

Proof. Since $\mathrm{pr}_1^{(1)} \in C$ and C is C-stable, we clearly have $\langle \mathrm{pr}_1^{(1)} \rangle \subseteq C$. By Lemma 3.2(ii), we also have $f = \mathrm{pr}_1^{(1)} * f \in \langle \mathrm{pr}_1^{(1)} \rangle_C$ for every $f \in C$, so $C \subseteq \langle \mathrm{pr}_1^{(1)} \rangle_C$. □

4. Linear stability over finite fields

In this section we consider classes of operations on an arbitrary finite field and their right and left stability under clones of linear functions. Assume that $A = \mathrm{GF}(q)$, a finite field of order $q = p^m$, with p prime.

Definition 4.1. It is well known that every n-ary operation on A is represented by a unique polynomial over $\mathrm{GF}(q)$ in n variables wherein no variable appears with an exponent greater than $q - 1$. We call such polynomials *reduced polynomials.* A reduced polynomial can be written as

$$\sum_{(a_1,\dots,a_n) \in \{0,\dots,q-1\}^n} \alpha_{(a_1,\dots,a_n)} \prod_{i \in \{1,\dots,n\}} x_i^{a_i}, \tag{1}$$

where each *coefficient* $\alpha_{(a_1,\dots,a_n)}$ is an element of $\mathrm{GF}(q)$. We will use the shorthand $\alpha_{\mathbf{a}} x^{\mathbf{a}}$ to designate the *monomial* $\alpha_{(a_1,\dots,a_n)} \prod_{i \in \{1,\dots,n\}} x_i^{a_i}$ with $\mathbf{a} = (a_1, \dots, a_n)$. A monomial with coefficient 1 is called *monic.* The *degree* of a monomial $\alpha_{\mathbf{a}} x^{\mathbf{a}}$ is $\sum_{i=1}^n a_i$. The *degree* of a polynomial p, denoted $\deg(p)$, is the maximum of the degrees of its monomials with a nonzero coefficient; we agree that $\deg(0) := 0$. In general, when we speak of the monomials of a polynomial, we mean the monomials with a nonzero coefficient. As is usual when writing polynomials, we may omit coefficients equal to 1, and we may omit monomials with coefficient 0. Without any risk of confusion, we will denote functions by reduced polynomials.

The *degree* of an operation f, denoted $\deg(f)$, is the degree of the unique reduced polynomial representing f. For $k \in \mathbb{N}$, denote by D_k the set of all operations on A of degree at most k. Clearly, these sets constitute an infinite ascending chain $\mathsf{D}_0 \subset \mathsf{D}_1 \subset \mathsf{D}_2 \subset \cdots$ whose union is the set $\mathcal{O}_A$ of all operations on A. In particular, D_0 is the set of all constant operations, and D_1 is the set of all *linear* operations.[1] We shall also use the symbol L to denote the set D_1. The set L is a clone on A; in fact, it is a maximal clone according to Rosenberg's classification [17].

Proposition 4.2. *For every $k \in \mathbb{N}$, the set D_k is L-stable.*

Proof. Noting that the clone L is generated by $\{x_1 + x_2\} \cup \{\, cx_1 \mid c \in A \,\} \cup \{\, c \mid c \in A \,\}$, we apply Lemmata 3.3 and 3.2. The stability under left composition with L follows from the fact that for any $f, g \in \mathsf{D}_k$ and any $c \in A$ we have $c(f) = c \in \mathsf{D}_0 \subseteq \mathsf{D}_k$, $cx_1(f) = c \cdot f \in \mathsf{D}_k$, and $(x_1 + x_2)(f, g) = f + g \in \mathsf{D}_k$. As for the right stability, note that D_k is minor-closed because the formation of minors does not increase the degree of functions, and that for any $f \in \mathsf{D}_k$ and for any $c \in A$, it holds that $f * c$, $f * cx_1$, and $f * (x_1 + x_2)$ are members of D_k. □

[1] Strictly speaking, operations of degree at most 1 are *affine* in the sense of linear algebra. We go along with the term *linear* that is common in the context of clone theory and especially in the theory of Boolean functions.

Proposition 4.3. *The empty set $\emptyset$ and the set $\mathcal{O}_A$ of all operations on A are* L*-stable.*

Proof. Trivial. □

Lemma 4.4. *Every nonempty* L*-stable class contains all constant functions.*

Proof. Let K be a nonempty L-stable class. Since L contains all projections of any arity, $K\mathsf{L}$ contains functions of any arity, and so does K because $K\mathsf{L} \subseteq K$. Note that for any $g_1, \dots, g_n \in \mathcal{O}_A^{(m)}$, it holds that $\mathrm{c}_b^{(n)}(g_1, \dots, g_n) = \mathrm{c}_b^{(m)}$. Since all constant functions are members of L and K contains functions of any arity, it follows that $\mathsf{L}K$ contains all constant functions, and so does K because $\mathsf{L}K \subseteq K$. □

Lemma 4.5. *For any $k \in \mathbb{N}$, $\langle x_1 x_2 \dots x_k \rangle_{\mathsf{L}} = \mathsf{D}_k$.*

Proof. Clearly $x_1 x_2 \dots x_k \in \mathsf{D}_k$ and D_k is L-stable by Proposition 4.2, so we have $\langle x_1 x_2 \dots x_k \rangle_{\mathsf{L}} \subseteq \mathsf{D}_k$. By identification of variables, permutation of variables, and substitution of constant 1 for variables, we obtain every monic monomial of degree at most k. By taking the sum of monic monomials of degree at most k, with suitable coefficients, we can obtain any polynomial of degree at most k, in other words, by composing a suitable linear function with functions represented by monic monomials of degree at most k, we obtain any function of degree at most k. Therefore, $\mathsf{D}_k \subseteq \langle x_1 x_2 \dots x_k \rangle_{\mathsf{L}}$. □

Lemma 4.6. *If the reduced polynomial of $f \colon A^n \to A$ has degree k, then $\langle f \rangle_{\mathsf{L}} = \mathsf{D}_k$.*

Proof. Let p be the reduced polynomial representing f as in (1). Let $\mathbf{u} = (u_1, \dots, u_n) \in \{0, \dots, q-1\}^n$ be such that $\alpha_{\mathbf{u}} x^{\mathbf{u}}$ has degree k and $\alpha_{\mathbf{u}} \neq 0$. We may assume that $\alpha_{\mathbf{u}} = 1$, because by composing f from the left by $\alpha_{\mathbf{u}}^{-1} x_1$, which belongs to L, we obtain a function in $\langle f \rangle_{\mathsf{L}}$ that has the same monomials as f but with coefficients multiplied by $\alpha_{\mathbf{u}}^{-1}$.

Let $U := \{\, i \in [n] \mid u_i \neq 0 \,\}$. By substituting 0 for the variables x_i with $i \in [n] \setminus U$, we obtain a function f' in $\langle f \rangle_{\mathsf{L}}$ with reduced polynomial p' such that p' has degree k and contains only variables x_i with $i \in U$, and $\alpha_{\mathbf{u}} x^{\mathbf{u}}$ is a monomial of degree k in p'. We may consider the function f' in place of f and assume, without loss of generality, that $U = [n]$.

Let $\{B_1, \dots, B_n\}$ be a partition of $[k]$ in n parts such that $|B_j| = u_j$ for all $j \in [n]$. For $j \in [n]$, let $g_j = \sum_{i \in B_j} x_i$. Note that $g_j \in \mathsf{L}$. Consider the function $h := f(g_1, \dots, g_n)$, which is in $\langle f \rangle_{\mathsf{L}}$. For every $\mathbf{a} \in \{0, \dots, q-1\}^n$ with $\sum_{i=1}^n a_i \leq k$, the expansion of the product $\prod_{i=1}^n g_i^{a_i}$ results in a polynomial of degree at most k in which no monomial contains all variables $x_1, \dots, x_k$, with the exception of $\mathbf{a} = \mathbf{u}$, for which the expansion yields a polynomial in which one of the monomials is $x_1 \dots x_k$ and the other monomials do not contain all variables $x_1, \dots, x_k$. Consequently, $h = x_1 \dots x_k + h'$ where h' is a polynomial in variables $x_1, \dots, x_k$ in which no monomial contains all variables $x_1, \dots, x_k$.

Now, let us define a sequence of functions $h_0, \dots, h_k$ recursively as follows: $h_0 := h$. For $i = 1, \dots, k$, let $h_i := h_{i-1} - h_{i-1}(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_k)$. We have $h_i \in \langle h_{i-1} \rangle_{\mathsf{L}}$. It is easy to see that the polynomial of h_i can be obtained from the polynomial of h_{i-1} by removing all monomials in which x_i does not occur. Consequently, $x_1 \dots x_k = h_k \in \langle h_{k-1} \rangle_{\mathsf{L}} \subseteq \langle h_{k-2} \rangle_{\mathsf{L}} \subseteq \dots \subseteq \langle h_0 \rangle_{\mathsf{L}} \subseteq \langle f \rangle_{\mathsf{L}}$. Now it follows from Lemma 4.5 that $\mathsf{D}_k = \langle x_1 \dots x_k \rangle_{\mathsf{L}} \subseteq \langle f \rangle_{\mathsf{L}} \subseteq \mathsf{D}_k$. □

Lemma 4.7. *Let $K \subseteq \mathcal{O}_A$, $K \neq \emptyset$. If the set $\{\, \deg(f) \mid f \in K \,\}$ has a maximum m, then $\langle K \rangle_{\mathsf{L}} = \mathsf{D}_m$. Otherwise $\langle K \rangle_{\mathsf{L}} = \mathcal{O}_A$.*

Proof. If said maximum m exists, we have $K \subseteq \mathsf{D}_m$ and there exists a $g \in K$ with $\deg(g) = m$. Since D_m is L-stable by Lemma 4.2, we have $\langle K \rangle_{\mathsf{L}} \subseteq \mathsf{D}_m$. Lemma 4.6 implies

$$\mathsf{D}_m = \mathsf{D}_{\deg(g)} \subseteq \bigcup_{f \in K} \mathsf{D}_{\deg(f)} = \bigcup_{f \in K} \langle f \rangle_{\mathsf{L}} \subseteq \langle K \rangle_{\mathsf{L}} \subseteq \mathsf{D}_m.$$

Otherwise there is no finite upper bound on the degrees of the members of K. Then for every $i \in \mathbb{N}$, there exists an $f_i \in K$ with $\deg(f_i) \geq i$. Now we have

$$\mathcal{O}_A = \bigcup_{i \in \mathbb{N}} \mathsf{D}_i \subseteq \bigcup_{i \in \mathbb{N}} \mathsf{D}_{\deg(f_i)} = \bigcup_{i \in \mathbb{N}} \langle f_i \rangle_{\mathsf{L}} \subseteq \langle K \rangle_{\mathsf{L}} \subseteq \mathcal{O}_A. \qquad \square$$

Theorem 4.8. *The* L*-stable classes are* $\mathcal{O}_A$*,* D_k*, and* $\emptyset$*, for* $k \in \mathbb{N}$*.*

Proof. The classes mentioned in the statement are L-stable by Propositions 4.2 and 4.3. Lemma 4.7 implies that there are no further L-stable classes. $\square$

5. Boolean functions

Definition 5.1. Operations on $\{0,1\}$ are called *Boolean functions.* The class of all Boolean functions is denoted by Ω.

Definition 5.2. By particularizing Definition 4.1 to the two-element field, we obtain that every Boolean function is represented by a unique *multilinear polynomial* over the two-element field, i.e., a polynomial with coefficients in GF(2) in which no variable appears with an exponent greater than 1. Since the coefficients come from the set $\{0,1\}$, every monomial with a nonzero coefficient is monic. The unique multilinear polynomial representing a Boolean function f is known as the *Zhegalkin polynomial* of f, and it can be written as

$$\sum_{S \in M_f} x_S, \tag{2}$$

where x_S is a shorthand for $\prod_{i \in S} x_i$ and $M_f \subseteq \mathcal{P}([n])$. Note that $x_\emptyset = 1$ and $\sum_{S \in \emptyset} x_S = 0$. The terms x_S with $S \neq \emptyset$ are called *monomials.* If $\emptyset \in M_f$, then we say that f has *constant term* 1; otherwise f has *constant term* 0. Without any risk of confusion, we will denote Boolean functions by their Zhegalkin polynomials, and we refer to the set M_f as the *set of monomials* of f.

Definition 5.3. Some well-known Boolean functions are defined in Table 1: modulo-2 addition $+$, conjunction $\wedge$, disjunction $\vee$, triple sum $\oplus_3$, median μ. Their Zhegalkin polynomials are the following:

$$\begin{aligned}
&x_1 + x_2, && \\
&x_1 \wedge x_2 = x_1 x_2, && \oplus_3(x_1, x_2, x_3) = x_1 + x_2 + x_3, \\
&x_1 \vee x_2 = x_1 x_2 + x_1 + x_2, && \mu(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3.
\end{aligned}$$

Definition 5.4. For $a, b \in \{0,1\}$, let

$$\Omega_{a*} := \{\, f \in \Omega \mid f(0, \dots, 0) = a \,\},$$
$$\Omega_{*b} := \{\, f \in \Omega \mid f(1, \dots, 1) = b \,\},$$

and let $\Omega_{ab} := \Omega_{a*} \cap \Omega_{*b}$. Furthermore, define

$$\Omega_{=} := \{\, f \in \Omega \mid f(0, \dots, 0) = f(1, \dots, 1) \,\},$$
$$\Omega_{\neq} := \{\, f \in \Omega \mid f(0, \dots, 0) \neq f(1, \dots, 1) \,\},$$

that is, $\Omega_{=} = \Omega_{00} \cup \Omega_{11}$ and $\Omega_{\neq} = \Omega_{01} \cup \Omega_{10}$.

Clearly $\Omega_{0*} \cap \Omega_{1*} = \emptyset$ and $\Omega_{0*} \cup \Omega_{1*} = \Omega$; similarly, $\Omega_{*0} \cap \Omega_{*1} = \emptyset$ and $\Omega_{*0} \cup \Omega_{*1} = \Omega$, and $\Omega_{=} \cap \Omega_{\neq} = \emptyset$ and $\Omega_{=} \cup \Omega_{\neq} = \Omega$. It is easy to see that Ω_{a*} is the class of all Boolean functions with constant term a.

x	y	$x+y$	$x \wedge y$	$x \vee y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	0	1	1

x	y	z	$\oplus_3(x, y, z)$	$\mu(x, y, z)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Table 1. Well-known Boolean functions

Definition 5.5. For $a \in \{0,1\}$, a Boolean function f is *a-preserving* if $f(a, \dots, a) = a$. A function is *constant-preserving* if it is both 0- and 1-preserving. We denote the classes of all 0-preserving, of all 1-preserving, and of all constant-preserving functions by T_0, T_1, and T_{c}, respectively. Note that $\mathsf{T}_{\mathrm{c}} = \mathsf{T}_0 \cap \mathsf{T}_1$. It follows from the definitions that $\mathsf{T}_0 = \Omega_{0*}$, $\mathsf{T}_1 = \Omega_{*1}$, and $\mathsf{T}_{\mathrm{c}} = \Omega_{01}$.

Remark 5.6. The reason why we have introduced multiple notation for the classes $\mathsf{T}_0 = \Omega_{0*}$ and $\mathsf{T}_1 = \Omega_{*1}$ is to facilitate writing certain statements in a parameterized form and to make reference, as the case may be, to either the classes Ω_{a*} ($a \in \{0,1\}$), Ω_{*b} ($b \in \{0,1\}$), or T_a ($a \in \{0,1\}$).

Definition 5.7. The *parity* of a Boolean function f, denoted $\operatorname{par}(f)$, is a number, either 0 or 1, which is given by

$$\operatorname{par}(f) := |M_f \setminus \{\emptyset\}| \bmod 2.$$

We call a function *even* or *odd* if its parity is 0 or 1, respectively. Note that $\Omega_{=}$ and $\Omega_{\neq}$ are precisely the classes of even and odd functions, respectively.

Definition 5.8. The set $\{0,1\}$ is endowed with the natural order $\leq$, with $0 < 1$, which induces the componentwise order, also denoted by $\leq$, on the Cartesian power $\{0,1\}^n$: for $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \{0,1\}^n$, $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$ for all $i \in [n]$.

A Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *monotone* if $f(\mathbf{a}) \leq f(\mathbf{b})$ whenever $\mathbf{a} \leq \mathbf{b}$. We denote by M the class of all monotone Boolean functions.

Definition 5.9. For $a \in \{0,1\}$, let $\overline{a}$ denote the *negation* of a, that is, $\overline{a} := 1 - a$. For any $f \in \Omega^{(n)}$, denote by $\overline{f}$ the *negation* of f, that is, the function $\overline{f}\colon \{0,1\}^n \to \{0,1\}$ with $\overline{f}(\mathbf{a}) = \overline{f(\mathbf{a})}$ for all $\mathbf{a} \in \{0,1\}^n$. For $C \subseteq \Omega$, let $\overline{C} := \{\, \overline{f} \mid f \in C \,\}$.

A function f is *self-dual* if $f(a_1, \dots, a_n) = \overline{f(\overline{a}_1, \dots, \overline{a}_n)}$ for all $a_1, \dots, a_n \in \{0,1\}$. A function f is *reflexive* (or *self-anti-dual*) if $f(a_1, \dots, a_n) = f(\overline{a}_1, \dots, \overline{a}_n)$ for all $a_1, \dots, a_n \in \{0,1\}$. We denote by S the class of all self-dual functions. Let $\mathsf{S}_{\mathrm{c}} := \mathsf{S} \cap \mathsf{T}_{\mathrm{c}}$ and $\mathsf{SM} := \mathsf{S} \cap \mathsf{M}$, the classes of constant-preserving self-dual functions and monotone self-dual functions, respectively.

Definition 5.10. By particularizing the definition of degree (see Definition 4.1) to monomials and polynomials over $\mathrm{GF}(2)$, we obtain that the *degree* of a monomial x_S is just $|S|$, and the *degree* of a Boolean function f is the size of the largest monomial in the Zhegalkin polynomial of f, i.e., $\deg(f) := \max_{S \in M_f} |S|$ for $f \neq 0$, and we agree that $\deg(0) := 0$. As before, for $k \in \mathbb{N}$, we denote by D_k the class of all Boolean functions of degree at most k. Clearly $\mathsf{D}_k \subsetneq \mathsf{D}_{k+1}$ for all $k \in \mathbb{N}$.

A Boolean function f is *linear* if $\deg(f) \leq 1$. We denote by L the class of all linear functions. Thus $\mathsf{L} = \mathsf{D}_1$. We also let

$$\mathsf{L}_0 := \mathsf{L} \cap \mathsf{T}_0 = \mathsf{L} \cap \Omega_{0*}, \qquad \mathsf{L}_1 := \mathsf{L} \cap \mathsf{T}_1 = \mathsf{L} \cap \Omega_{*1},$$
$$\mathsf{LS} := \mathsf{L} \cap \mathsf{S} = \mathsf{L} \cap \Omega_{\neq}, \qquad \mathsf{L}_\mathrm{c} := \mathsf{L} \cap \mathsf{T}_\mathrm{c} = \mathsf{L} \cap \Omega_{01}.$$

The equalities in the above definitions are clear by Remark 5.6, except for the equality $\mathsf{LS} = \mathsf{L} \cap \Omega_{\neq}$ which is easy to verify and also follows from Lemma 5.12 below.

Definition 5.11. Let f be an n-ary Boolean function. The *characteristic* of a set $S \subseteq [n]$ in f is a number, either 0 or 1, which is given by

$$\operatorname{ch}(S, f) := |\{\, A \in M_f \mid S \subsetneq A \,\}| \bmod 2.$$

The *characteristic rank* of f, denoted by $\chi(f)$, is the smallest integer m such that $\operatorname{ch}(S, f) = 0$ for all subsets $S \subseteq [n]$ with $|S| \geq m$. Clearly $\chi(f) \leq n$ because $\operatorname{ch}([n], f) = 0$.

For $k \in \mathbb{N}$, denote by X_k the class of all Boolean functions of characteristic rank at most k. For any $k \in \mathbb{N}$, we have $\mathsf{X}_k \subsetneq \mathsf{X}_{k+1}$. The inclusion is proper, as witnessed by the function $x_1 \dots x_{k+1} \in \mathsf{X}_{k+1} \setminus \mathsf{X}_k$. Moreover, for any $k \in \mathbb{N}$, we have $\mathsf{D}_k \subseteq \mathsf{X}_k$.

Reflexive and self-dual functions have a beautiful characterization in terms of the characteristic rank.

Lemma 5.12 (Selezneva, Bukhman [18, Lemmata 3.1, 3.5])**.**

(i) *A Boolean function f is reflexive if and only if $\chi(f) = 0$.*
(ii) *A Boolean function f is self-dual if and only if $f + x_1$ is reflexive.*
(iii) *A Boolean function f is self-dual if and only if f is odd and $\chi(f) = 1$.*

In other words, $\mathsf{X}_0 = \mathsf{X}_1 \cap \Omega_{=}$ is the class of all reflexive functions, $\mathsf{X}_1 \cap \Omega_{\neq}$ is the class of all self-dual functions, and X_1 is the class of all self-dual or reflexive functions.

Definition 5.13. Let Λ_c and V_c denote the classes of all conjunctions of arguments and of all disjunctions of arguments, respectively, that is,

$$\Lambda_\mathrm{c} := \{\, f \in \Omega^{(n)} \mid n \in \mathbb{N}_+,\ \emptyset \neq \{i_1, \dots, i_r\} \subseteq [n],\ f(a_1, \dots, a_n) = a_{i_1} \wedge \cdots \wedge a_{i_r} \,\},$$
$$\mathsf{V}_\mathrm{c} := \{\, f \in \Omega^{(n)} \mid n \in \mathbb{N}_+,\ \emptyset \neq \{i_1, \dots, i_r\} \subseteq [n],\ f(a_1, \dots, a_n) = a_{i_1} \vee \cdots \vee a_{i_r} \,\}.$$

Let I_c, I_0, I_1, and I^* denote the class of all projections, the class of all projections and constant 0 functions, the class of all projections and constant 1 functions, and the class of all projections and negated projections, respectively, that is,

$$\mathsf{I}_\mathrm{c} := \{\, \mathrm{pr}_i^{(n)} \mid i, n \in \mathbb{N}_+,\ 1 \leq i \leq n \,\},$$
$$\mathsf{I}_0 := \mathsf{I}_\mathrm{c} \cup \{\, \mathrm{c}_0^{(n)} \mid n \in \mathbb{N}_+ \,\},$$
$$\mathsf{I}_1 := \mathsf{I}_\mathrm{c} \cup \{\, \mathrm{c}_1^{(n)} \mid n \in \mathbb{N}_+ \,\},$$
$$\mathsf{I}^* := \mathsf{I}_\mathrm{c} \cup \overline{\mathsf{I}_\mathrm{c}}.$$

It was shown by Post [15] that there are a countably infinite number of clones of Boolean functions. In this paper, we will only need a handful of them, namely the clones Ω, T_0, T_1, T_c, M, S, S_c, SM, L, L_0, L_1, LS, L_c, Λ_c, V_c, I^*, I_0, I_1, and I_c that were defined above. The lattice of clones of Boolean functions, the so-called *Post's lattice,* is shown in Figure 1, and the above-mentioned clones are indicated in the diagram. In what follows, we will often make use of the following generating sets

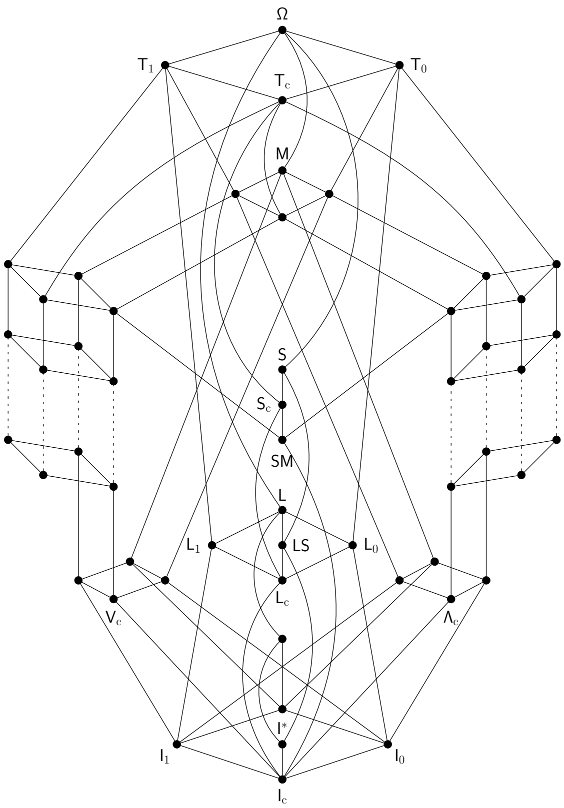

FIGURE 1. Post's lattice.

for some of these clones.

$$\begin{aligned}
\mathsf{\Omega} &= \langle x_1 x_2 + 1 \rangle, & \mathsf{S} &= \langle \mu,\, x_1 + 1 \rangle, & \mathsf{SM} &= \langle \mu \rangle, & \mathsf{L} &= \langle x_1 + x_2, 1 \rangle, \\
\mathsf{LS} &= \langle \oplus_3,\, x_1 + 1 \rangle, & \mathsf{L_c} &= \langle \oplus_3 \rangle, & \mathsf{\Lambda_c} &= \langle \wedge \rangle, & \mathsf{V_c} &= \langle \vee \rangle, \\
\mathsf{I}^* &= \langle x_1 + 1 \rangle, & \mathsf{I_0} &= \langle 0 \rangle, & \mathsf{I_1} &= \langle 1 \rangle, & \mathsf{I_c} &= \langle \emptyset \rangle.
\end{aligned}$$

Let us conclude this introductory section with a couple of lemmata that help us express sums and minors of Boolean functions in terms of their sets of monomials.

Lemma 5.14. *Let* $f\colon \{0,1\}^n \to \{0,1\}$ *and* $f\colon \{0,1\}^n \to \{0,1\}$. *Then* $M_{f+g} = M_f \bigtriangleup M_g$.

Proof. By adding the polynomials of f and g and by cancelling equal monomials (because we do addition modulo 2), we obtain

$$f + g = \sum_{S \in M_f} x_S + \sum_{S \in M_g} x_S = \sum_{S \in M_f \bigtriangleup M_g} x_S.$$

Consequently, $M_{f+g} = M_f \bigtriangleup M_g$ by the uniqueness of Zhegalkin polynomials. □

Lemma 5.15. *Let* $f\colon \{0,1\}^n \to \{0,1\}$ *and* $\sigma\colon [n] \to [m]$. *Then*

$$M_{f_\sigma} = \bigl\{\, S \subseteq [m] \bigm| |\{\, T \in M_f \mid \sigma(T) = S \,\}| \equiv 1 \pmod 2 \,\bigr\}.$$

Proof. A straightforward calculation using the definitions of minor and M_f (Definitions 2.5 and 5.2) shows that for all $a_1, \dots, a_m \in \{0,1\}$,

$$f_\sigma(a_1, \dots, a_m) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = \sum_{T \in M_f} \prod_{i \in T} a_{\sigma(i)} = \sum_{T \in M_f} \prod_{i \in \sigma(T)} a_i$$

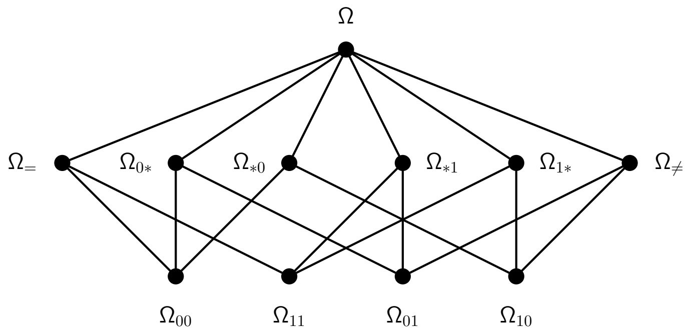

FIGURE 2. A block of eleven L_c-stable classes.

By cancelling pairs of summands corresponding to indices $T, T' \in M_f$ such that $\sigma(T) = \sigma(T')$, which are equal for any $a_1, \dots, a_m$, we get

$$\sum_{T \in M_f} \prod_{i \in \sigma(T)} a_i = \sum_{S \in M'} \prod_{i \in S} a_i,$$

where

$$M' = \bigl\{\, S \subseteq [m] \bigm| |\{\, T \in M_f \mid \sigma(T) = S \,\}| \equiv 1 \pmod 2 \,\bigr\}.$$

Consequently, $M_{f_\sigma} = M'$ by the uniqueness of Zhegalkin polynomials. □

6. L_c-STABLE CLASSES

We are now ready to state the main result of this paper, a complete description of the L_c-stable classes of Boolean functions.

Theorem 6.1. *The L_c-stable classes or, equivalently, the $(\mathsf{I}_\mathrm{c}, \mathsf{L}_\mathrm{c})$-stable classes are*

$$\begin{array}{lllll}
\Omega, & \Omega_{a*}, & \Omega_{*b}, & \Omega_{\approx}, & \Omega_{ab}, \\
\mathsf{D}_k, & \mathsf{D}_k \cap \Omega_{a*}, & \mathsf{D}_k \cap \Omega_{*b}, & \mathsf{D}_k \cap \Omega_{\approx}, & \mathsf{D}_k \cap \Omega_{ab}, \\
\mathsf{X}_k, & \mathsf{X}_k \cap \Omega_{a*}, & \mathsf{X}_k \cap \Omega_{*b}, & \mathsf{X}_k \cap \Omega_{\approx}, & \mathsf{X}_k \cap \Omega_{ab}, \\
\mathsf{D}_i \cap \mathsf{X}_j, & \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}, & \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*b}, & \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}, & \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}, \\
\mathsf{D}_0, & \mathsf{D}_0 \cap \Omega_{a*}, & \emptyset, & &
\end{array}$$

for $a, b \in \{0, 1\}$, $\approx \in \{=, \neq\}$, and $i, j, k \in \mathbb{N}_+$ with $i > j \geq 1$.

Several L_c-stable classes were known previously: the clones Ω, $\mathsf{S} = \mathsf{X}_1 \cap \Omega_{\neq}$, $\mathsf{L} = \mathsf{D}_1$, $\mathsf{T}_0 = \Omega_{0*}$, $\mathsf{T}_1 = \Omega_{*1}$, $\mathsf{T}_\mathrm{c} = \Omega_{01}$, $\mathsf{S}_\mathrm{c} = \mathsf{X}_1 \cap \Omega_{01}$, $\mathsf{L}_0 = \mathsf{D}_1 \cap \Omega_{0*}$, $\mathsf{L}_1 = \mathsf{D}_1 \cap \Omega_{*1}$, $\mathsf{LS} = \mathsf{D}_1 \cap \mathsf{X}_1 \cap \Omega_{\neq}$, $\mathsf{L}_\mathrm{c} = \mathsf{D}_1 \cap \Omega_{01}$, as well as the classes D_k for any $k \in \mathbb{N}$ and the class X_0 of reflexive (self-anti-dual) functions [4, pp. 29, 33]. The classes D_k for $k \in \mathbb{N}$ were also known to be L_0-stable [6, Example 1, p. 111].

In order to describe the structure of the lattice of L_c-stable classes, it is helpful to first look at the poset comprising the eleven classes Ω, $\Omega_=$, $\Omega_{\neq}$, Ω_{0*}, Ω_{1*}, Ω_{*0}, Ω_{*1}, Ω_{00}, Ω_{01}, Ω_{10}, Ω_{11} that is shown in Figure 2. It is noteworthy that the four minimal classes of this poset are pairwise disjoint, and that the six lower covers of Ω are precisely the unions of the six different pairs of minimal classes.

The lattice of all L_c-stable classes is shown in Figure 3. It has rather regular structure; it is isomorphic to the direct product of the 11-element poset of Figure 2 and the set $\{\, (i, j) \in (\mathbb{N}_+ \cup \{\infty\})^2 \mid i \geq j \geq 1 \,\}$ with the componentwise order, and a few additional elements near the bottom of the lattice. In order to avoid clutter, we have used some shorthand notation in Figure 3. The diagram includes multiple copies of the 11-element poset of Figure 2 (the shaded blocks) connected by thick triple lines. Each thick triple line between a pair of blocks represents eleven edges, each connecting a vertex of one poset to its corresponding vertex in the other poset. We have labeled in the diagram the meet-irreducible classes, as well as a few other classes of interest; the remaining classes are intersections of the meet-irreducible ones.

The remainder of this section is devoted to the proof of Theorem 6.1. The proof has two parts. First we observe that the classes listed in Theorem 6.1 are L_c-stable. Secondly, we need to show that there are no other L_c-stable classes.

To this end, we start with verifying that the classes of Theorem 6.1 are L_c-stable. Since intersections of L_c-stable classes are L_c-stable, it suffices to verify this for the meet-irreducible classes. With the help of the following lemma, we can further simplify the task of checking the stability under left and right composition with clones containing the triple sum. In fact L_c-stability is equivalent to $(\mathsf{I}_\mathrm{c}, \mathsf{L}_\mathrm{c})$-stability.

Lemma 6.2.

(i) *For any $f \in \Omega$, we have $f * \oplus_3 = \oplus_3(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3})$, where, for $i \in [3]$, $\sigma_i \colon [n] \to [n+2]$, $1 \mapsto i$, $j \mapsto j+2$ for $2 \leq j \leq n$.*

(ii) *Let $G \subseteq \Omega$, let $C_1 := \langle G \cup \{\oplus_3\} \rangle$, $C'_1 := \langle G \rangle$, and let C_2 be a clone containing $\oplus_3$. Then a class $F \subseteq \Omega$ is (C_1, C_2)-stable if and only if it is (C'_1, C_2)-stable.*

(iii) *The following are equivalent for a class $F \subseteq \Omega$.*
  (a) *F is L_c-stable.*
  (b) *F is $(\mathsf{I}_\mathrm{c}, \mathsf{L}_\mathrm{c})$-stable.*
  (c) *F is minor-closed and $f + g + h \in F$ whenever $f, g, h \in F$.*

Proof. (i) Let

$$A_i := \sigma_i(\{\, S \in M_f \mid 1 \in S \,\}) \qquad \text{for } i \in [3],$$
$$B := \sigma_1(\{\, S \in M_f \mid 1 \notin S \,\}) = \sigma_2(\{\, S \in M_f \mid 1 \notin S \,\}) = \sigma_3(\{\, S \in M_f \mid 1 \notin S \,\}).$$

Since the sets A_1, A_2, A_3, B are pairwise disjoint, their union equals their symmetric difference. Using the commutativity and associativity of the symmetric difference and the fact that $X \triangle X = \emptyset$ and $X \triangle \emptyset = X$ for any set X, we obtain

$$\begin{aligned} M_{f * \oplus_3} &= A_1 \cup A_2 \cup A_3 \cup B = A_1 \triangle A_2 \triangle A_3 \triangle B = A_1 \triangle A_2 \triangle A_3 \triangle B \triangle B \triangle B \\ &= (A_1 \triangle B) \triangle (A_2 \triangle B) \triangle (A_3 \triangle B) = (A_1 \cup B) \triangle (A_2 \cup B) \triangle (A_3 \cup B) \\ &= M_{f_{\sigma_1}} \triangle M_{f_{\sigma_2}} \triangle M_{f_{\sigma_3}} = M_{f_{\sigma_1} + f_{\sigma_2} + f_{\sigma_3}} = M_{\oplus_3(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3})}, \end{aligned}$$

that is, $f * \oplus_3 = \oplus_3(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3})$.

(ii) Since $C'_1 \subseteq C_1$, stability under right composition with C_1 implies stability under right composition with C'_1. Assume now that F is (C'_1, C_2)-stable. By Lemma 3.2, F is minor-closed and $f * g \in F$ whenever $f \in F$ and $g \in G$. Moreover, $f * \oplus_3 = \oplus_3(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3})$, where f_{σ_1}, f_{σ_2}, f_{σ_3} are the minors of f specified in part (i). Since F is minor-closed, we have $f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3} \in F$. By our assumption, $\oplus_3 \in C_2$, and since F is stable under left composition with C_2, it follows that $\oplus_3(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}) \in F$. It follows from Lemma 3.2 that F is stable under right composition with C_1.

(iii) Since $\mathsf{L}_\mathrm{c} = \langle \oplus_3 \rangle$, this is a consequence of part (ii) and Lemma 3.3. ☐

In view of Lemma 6.2(iii), our task is reduced to verifying that each one of the meet-irreducible classes shown in Figure 3, namely Ω, Ω_{0*}, Ω_{1*}, Ω_{*0}, Ω_{*1}, $\Omega_{=}$, $\Omega_{\neq}$, D_k, and X_k for $k \in \mathbb{N}$, is minor-closed and closed under triple sums of its members.

Lemma 6.3. *Ω is minor-closed and closed under triple sums of its members.*

Proof. Trivial. ☐

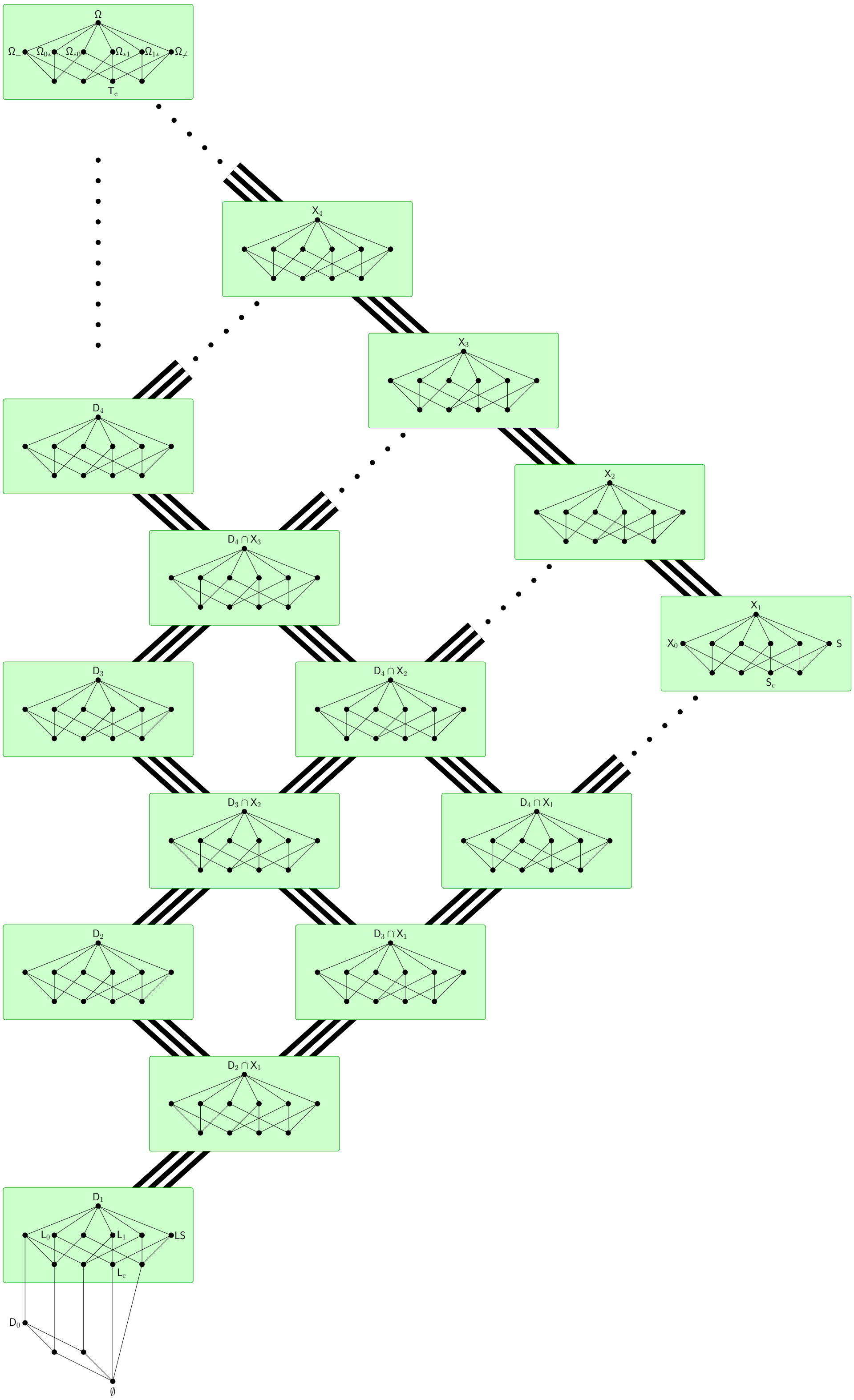

Figure 3. $\mathsf{L_c}$-stable classes.

Lemma 6.4. *Let $a, b \in \{0,1\}$.*

(i) *Ω_{a*} is minor-closed and closed under triple sums of its members.*
(ii) *Ω_{*b} is minor-closed and closed under triple sums of its members.*

Proof. (i) Let $f \in \Omega_{a*}^{(n)}$, and let $\sigma \colon [n] \to [m]$. We have $f_\sigma(0, \dots, 0) = f(0, \dots, 0) = a$, so $f_\sigma \in \Omega_{a*}$; thus Ω_{a*} is minor-closed. Let now $f, g, h \in \Omega_{a*}^{(n)}$. We have $(f + g + h)(0, \dots, 0) = f(0, \dots, 0) + g(0, \dots, 0) + h(0, \dots, 0) = a + a + a = a$; thus $f + g + h \in \Omega_{a*}$.

(ii) The proof is similar to that of part (i). $\square$

Lemma 6.5. *For $\approx \in \{=, \neq\}$, $\Omega_\approx$ is minor-closed and closed under triple sums of its members.*

Proof. We show first that $\Omega_\approx$ is minor-closed. Let $f \colon \{0,1\}^n \to \{0,1\}$ and $\sigma \colon [n] \to [m]$. For each $T \subseteq [m]$, let $Z_T := \{\, S \in M_f \mid \sigma(T) = S \,\}$; clearly the sets Z_T are pairwise disjoint and their union is M_f. By Lemma 5.15, $|Z_T|$ is odd if and only if $T \in M_{f_\sigma}$; thus for each $T \subseteq [m]$, there exists a number $k_T \in \mathbb{N}$ such that $|Z_T| = 2k_T + 1$ if $T \in M_{f_\sigma}$ and $|Z_T| = 2k_T$ if $T \notin M_{f_\sigma}$. Consequently,

$$\begin{aligned} |M_f| &= \sum_{T \subseteq [m]} |Z_T| = \sum_{\substack{T \subseteq [m] \\ T \in M_{f_\sigma}}} (2k_Z + 1) + \sum_{\substack{T \subseteq [m] \\ T \notin M_{f_\sigma}}} 2k_Z \\ &\equiv \sum_{\substack{T \subseteq [m] \\ T \in M_{f_\sigma}}} 1 + \sum_{\substack{T \subseteq [m] \\ T \notin M_{f_\sigma}}} 0 = |M_{f_\sigma}| \pmod{2}. \end{aligned}$$

Note also that $Z_\emptyset = \{\emptyset\}$; thus $\emptyset \in M_{f_\sigma}$ if and only if $\emptyset \in M_f$. It follows that $\operatorname{par}(f_\sigma) = \operatorname{par}(f)$. Therefore $f_\sigma \in \Omega_\approx$ if and only if $f \in \Omega_\approx$, that is, $\Omega_\approx$ is minor-closed.

We now show that $\Omega_\approx$ is closed under triple sums of its members. Let $f, g, h \in \Omega_\approx^{(n)}$. By definition, it holds that $\operatorname{par}(f) = |M_f \setminus \{\emptyset\}| \bmod 2 = a$, $\operatorname{par}(g) = |M_g \setminus \{\emptyset\}| \bmod 2 = a$, $\operatorname{par}(h) = |M_h \setminus \{\emptyset\}| \bmod 2 = a$, where $a = 0$ if $\approx$ is $=$ and $a = 1$ otherwise. Then $M_{f+g+h} = M_f \triangle M_g \triangle M_h$ by Lemma 5.14, so $M_{f+g+h} \setminus \{\emptyset\} = (M_f \setminus \{\emptyset\}) \triangle (M_g \setminus \{\emptyset\}) \triangle (M_h \setminus \{\emptyset\})$, which implies $\operatorname{par}(f + g + h) = |M_{f+g+h} \setminus \{\emptyset\}| \bmod 2 = a + a + a \bmod 2 = a$. Therefore $f + g + h \in \Omega_\approx$. $\square$

Lemma 6.6. *For $k \in \mathbb{N}$, D_k is minor-closed and closed under sums of its members.*

Proof. Let $f \in \mathsf{D}_k^{(n)}$, and let $\sigma \colon [n] \to [m]$. Let $T \in M_{f_\sigma}$ be such that $|T| = \deg f_\sigma$. By Lemma 5.15, there exists $U \in M_f$ such that $\sigma(U) = T$. We must have $|T| \leq |U|$, so $\deg f_\sigma = |T| \leq |U| \leq \deg f \leq k$; therefore $f_\sigma \in \mathsf{D}_k$, so D_k is minor-closed.

Let now $f, g \in \mathsf{D}_k^{(n)}$. Since $M_{f+g} = M_f \triangle M_g$ by Lemma 5.14, we have $\deg(f + g) = \max_{S \in M_{f+g}} |S| \leq \max(\deg(f), \deg(g)) \leq k$, so $f + g \in \mathsf{D}_k$. $\square$

Lemma 6.7. *For $k \in \mathbb{N}$, X_k is minor-closed.*

Proof. In view of Remark 2.7, it is sufficient to consider closure under minors formed via injective maps or identification maps (see Definition 2.5). Let $f \in \mathsf{X}_k^{(n)}$.

Consider first an injective minor formation map $\sigma \colon [n] \to [m]$. It is easy to verify that $M_{f_\sigma} = \{\, \sigma(A) \mid A \in M_f \,\}$. Let $S \subseteq [m]$ with $|S| \geq k$. If $S \nsubseteq \operatorname{Im} \sigma$, then there is clearly no $A \in M_{f_\sigma}$ such that $S \nsubseteq A$; hence $\operatorname{ch}(S, f_\sigma) = 0$. If $S \subseteq \operatorname{Im} \sigma$, then $\operatorname{ch}(S, f_\sigma) = \operatorname{ch}(\sigma^{-1}(S), f) = 0$ because $|\sigma^{-1}(S)| = |S| \geq k$ and $f \in \mathsf{X}_k$. We conclude that $f_\sigma \in \mathsf{X}_k$.

Consider now an identification map $\sigma_{ij} \colon [n] \to [n-1]$ for some $i, j \in [n]$ with $i < j$. Let $S \subseteq [n-1]$ with $|S| \geq k$. Let $H := \{\, A \in M_f \mid S \subsetneq \sigma_{ij}(A) \,\}$. We claim that $|H|$ is even.

If $i \notin S$, then the only subset of $[n]$ mapped onto S by σ_{ij} is $\sigma_{ij}^{-1}(S)$, and any proper superset of $\sigma_{ij}^{-1}(S)$ is mapped to a proper superset of S. Therefore $H = \{\, A \in M_f \mid \sigma_{ij}^{-1}(S) \subsetneq A \,\}$. Since $|\sigma^{-1}(S)| = |S| \geq k$ and $f \in \mathsf{X}_k$, it follows that $|H|$ is even.

If $i \in S$, then there are three subsets of $[n]$ that are mapped onto S by σ_{ij}: $T_1 := \sigma_{ij}^{-1}(S)$, $T_2 := \sigma_{ij}^{-1}(S) \setminus \{i\}$, and $T_3 := \sigma_{ij}^{-1}(S) \setminus \{j\}$. The proper supersets of T_1, T_2, and T_3 are mapped to proper supersets of S, with the exception of the set $T_1 = \sigma_{ij}^{-1}(S)$, which is a proper superset of both T_2 and T_3 but $\sigma_{ij}(T_1) = S$. Since $|\sigma_{ij}^{-1}(S)| = |S| + 1 \geq k + 1$ and $f \in \mathsf{X}_k$, it follows that the sets $U_i := \{\, A \in M_f \mid T_i \subsetneq A \,\}$, $1 \leq i \leq 3$, have even cardinality. Let $U'_i := U_i \setminus \{T_1\}$, $i \in \{2, 3\}$, and observe that $U_1 = U'_2 \cap U'_3$. We have $H = U'_2 \cup U'_3$, so

$$|H| = |U'_2| + |U'_3| - |U_1| = \begin{cases} |U_2| + |U_3| - |U_1|, & \text{if } T_1 \notin M_f, \\ |U_2| + |U_3| - |U_1| - 2, & \text{if } T_1 \in M_f. \end{cases}$$

In either case, $|H|$ is even.

For any subset $A \subseteq [n-1]$, let $Z_A := \{\, T \subseteq M_f \mid \sigma_{ij}(T) = A \,\}$. By definition, we have $H = \bigcup_{S \subsetneq A \subseteq [n-1]} Z_A$. The sets Z_A are clearly pairwise disjoint, so $|H| = \sum_{S \subsetneq A \subseteq [n-1]} |Z_A|$. Since $|H|$ is even, there must be an even number of sets A with $S \subsetneq A \subseteq [n-1]$ such that $|Z_A|$ is odd. It follows from Lemma 5.15 that $\mathrm{ch}(S, f_{\sigma_{ij}}) = |\{\, A \in M_{f_{\sigma_{ij}}} \mid S \subsetneq A \,\}| \bmod 2 = 0$, and we conclude that $f_{\sigma_{ij}} \in \mathsf{X}_k$. ☐

Lemma 6.8. *Let $k \in \mathbb{N}$. For any $f, g \in \mathsf{X}_k^{(n)}$, we have $f + g \in \mathsf{X}_k$.*

Proof. Write $h := f + g$. We have $M_h = M_f \triangle M_g$ by Lemma 5.14, and for any $S \subseteq [n]$, it holds that

$$\begin{aligned} \{\, A \in M_h \mid S \subsetneq A \,\} &= \{\, A \in M_f \triangle M_g \mid S \subsetneq A \,\} \\ &= \{\, A \in M_f \mid S \subsetneq A \,\} \triangle \{\, A \in M_g \mid S \subsetneq A \,\}. \end{aligned}$$

By our assumption, for any $S \subseteq [n]$ with $|S| \geq k$, we have

$$\begin{aligned} |\{\, A \in M_f \mid S \subsetneq A \,\}| \bmod 2 = \mathrm{ch}(S, f) = 0, \\ |\{\, A \in M_g \mid S \subsetneq A \,\}| \bmod 2 = \mathrm{ch}(S, g) = 0. \end{aligned}$$

Since the symmetric difference of sets of even cardinality is again of even cardinality, it follows that $\mathrm{ch}(S, h) = |\{\, A \in M_h \mid S \subsetneq A \,\}| \bmod 2 = 0$ for any $S \subseteq [n]$ with $|S| \geq k$. Therefore $h \in \mathsf{X}_k$. ☐

Proposition 6.9. *The classes listed in Theorem 6.1 are L_c-stable.*

Proof. According to Lemmata 6.3, 6.4, 6.5, 6.6, 6.7, and 6.8, each of the classes Ω, Ω_{0*}, Ω_{1*}, Ω_{*0}, Ω_{*1}, $\Omega_{=}$, $\Omega_{\neq}$, D_k, and X_k for $k \in \mathbb{N}$ is minor-closed and closed under triple sums of its members, so by Lemma 6.2(iii), each is L_c-stable. It follows that the remaining classes listed in Theorem 6.1, being intersections of the above classes, are also L_c-stable. ☐

It remains to show that the classes listed in Theorem 6.1 are the only L_c-stable classes. To this end, we are going to verify that any set of Boolean functions generates exactly what is suggested by Figure 3. More precisely, we prove that each class K is generated by any subset of K that is not contained in any proper subclass fo K, i.e., the subset contains for each proper subclass C of K an element in $K \setminus C$. If each proper subclass is contained in a lower cover of K, then it suffices to consider the lower covers of K. We begin with some helpful lemmata.

Lemma 6.10. *For any $F \subseteq \Omega$, we have $f \in \langle F \rangle_{\mathsf{L_c}}$ if and only if f is the sum of an odd number of minors of members of F, i.e., $f = \sum_{i=1}^{2k+1} (g_i)_{\sigma_i}$ for some $k \in \mathbb{N}$, $g_i \in F$, $\sigma_i \colon [n_i] \to [n]$, where $n_i := \operatorname{ar}(g_i)$ and $n := \operatorname{ar}(f)$ $(1 \leq i \leq 2k+1)$.*

Proof. "$\Leftarrow$": Clear because $\langle F \rangle_{\mathsf{L_c}}$ is closed under minors and triple sums and hence under any odd sums of its members by Lemma 6.2(iii).

"$\Rightarrow$": By Lemma 6.2(iii), $\langle F \rangle_{\mathsf{L_c}}$ is the set obtained by a finite number of the following construction steps:

(1) Every $f \in F$ is a member of $\langle F \rangle_{\mathsf{L_c}}$.
(2) If $f \in \langle F \rangle_{\mathsf{L_c}}$, $\operatorname{ar}(f) = n$, and $\sigma \colon [n] \to [m]$ for some $m \in \mathbb{N}_+$, then $f_\sigma \in \langle F \rangle_{\mathsf{L_c}}$.
(3) If $f, g, h \in \langle F \rangle_{\mathsf{L_c}}$, all of arity $n \in \mathbb{N}_+$, then $f + g + h \in \langle F \rangle_{\mathsf{L_c}}$.

We will show by induction on the construction that every $f \in \langle F \rangle_{\mathsf{L_c}}$ is an odd sum of minors of members of F. This obviously holds for every $f \in F$: $f = \sum_{i=1}^{1} f_{\mathrm{id}}$. Assume $f = \sum_{i=1}^{2k+1} (g_i)_{\sigma_i}$ for some $g_i \in F$ and $\sigma_i \colon [n_i] \to [n]$ $(1 \leq i \leq 2k+1)$. Then for any $\tau \colon [n] \to [m]$, we have

$$f_\tau = \Bigl(\sum_{i=1}^{2k+1} (g_i)_{\sigma_i} \Bigr)_\tau = \sum_{i=1}^{2k+1} ((g_i)_{\sigma_i})_\tau = \sum_{i=1}^{2k+1} (g_i)_{\tau \circ \sigma_i},$$

where the second and the third equalities hold by Lemmata 2.8 and 2.6, respectively. Finally, assume that $f = \sum_{i=1}^{2k+1} (f_i)_{\sigma_i}$, $g = \sum_{i=1}^{2\ell+1} (g_i)_{\tau_i}$, $h = \sum_{i=1}^{2m+1} (h_i)_{\rho_i}$ for some $f_i, g_i, h_i \in F$, $\sigma_i \colon [\operatorname{ar}(f_i)] \to [n]$, $\tau_i \colon [\operatorname{ar}(g_i)] \to [n]$, $\rho_i \colon [\operatorname{ar}(h_i)] \to [n]$. Then

$$f + g + h = \sum_{i=1}^{2k+1} (f_i)_{\sigma_i} + \sum_{i=1}^{2\ell+1} (g_i)_{\tau_i} + \sum_{i=1}^{2m+1} (h_i)_{\rho_i},$$

which is an odd sum of minors of members of F. $\square$

Lemma 6.11. *Assume that C is an $\mathsf{L_c}$-stable class and $\langle F \rangle_{\mathsf{L_c}} = C$. Then $\overline{C}$ is $\mathsf{L_c}$-stable and $\langle \overline{F} \rangle_{\mathsf{L_c}} = \overline{C}$.*

Proof. Assume that $\langle F \rangle_{\mathsf{L_c}} = C$. Then $\overline{C}$ is $\mathsf{L_c}$-stable because for all n-ary $f + 1, g + 1, h + 1 \in \overline{C}$, we have $f, g, h \in C$ and hence $(f + 1) + (g + 1) + (h + 1) = (f + g + h) + 1 \in \overline{C}$, and for any $\sigma \colon [n] \to [m]$, we have, by Lemma 2.8, $(f + 1)_\sigma = f_\sigma + 1_\sigma = f_\sigma + 1 \in \overline{C}$.

In order to show that $\overline{C}$ is generated by $\overline{F}$, let $f + 1 \in \overline{C}$. Then $f \in C$, and by Lemma 6.10, $f = \sum_{i=1}^{2k+1} (g_i)_{\sigma_i}$ for some $g_i \in F$ and some minor formation map σ_i $(1 \leq i \leq 2k+1)$. Consequently, $f + 1 = \sum_{i=1}^{2k+1} ((g_i)_{\sigma_i} + 1) = \sum_{i=1}^{2k+1} ((g_i)_{\sigma_i} + 1_{\sigma_i}) = \sum_{i=1}^{2k+1} (g_i + 1)_{\sigma_i}$ by Lemma 2.8. Since each $g_i + 1$ is in $\overline{F}$, Lemma 6.10 implies that $f \in \langle \overline{F} \rangle_{\mathsf{L_c}}$. $\square$

Proposition 6.12.

(i) $\langle \emptyset \rangle_{\mathsf{L_c}} = \emptyset$.
(ii) *For any $f \in \mathsf{D}_0 \cap \Omega_{0*}$, we have $\langle f \rangle_{\mathsf{L_c}} = \mathsf{D}_0 \cap \Omega_{0*}$.*
(iii) *For any $f \in \mathsf{D}_0 \cap \Omega_{1*}$, we have $\langle f \rangle_{\mathsf{L_c}} = \mathsf{D}_0 \cap \Omega_{1*}$.*
(iv) *For any $f, g \in \mathsf{D}_0$ such that $f \notin \Omega_{0*}$, $g \notin \Omega_{1*}$, we have $\langle f, g \rangle_{\mathsf{L_c}} = \mathsf{D}_0$.*

Proof. (i) Obvious.

(ii) The function f is a constant 0 function of some arity. We obtain any constant 0 function by identifying arguments or introducing fictitious arguments. Therefore $\mathsf{D}_0 \cap \Omega_{0*} \subseteq \langle f \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_0 \cap \Omega_{0*}$.

(iii) Follows from part (ii) by Lemma 6.11 because $\overline{\mathsf{D}_0 \cap \Omega_{0*}} = \mathsf{D}_0 \cap \Omega_{1*}$.

(iv) Since Ω_{0*} and Ω_{1*} partition Ω, it follows that $f \in \mathsf{D}_0 \cap \Omega_{1*}$ and $g \in \mathsf{D}_0 \cap \Omega_{0*}$. By parts (ii) and (iii), $\mathsf{D}_0 = (\mathsf{D}_0 \cap \Omega_{0*}) \cup (\mathsf{D}_0 \cap \Omega_{1*}) = \langle g \rangle_{\mathsf{L_c}} \cup \langle f \rangle_{\mathsf{L_c}} \subseteq \langle f, g \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_0$. □

Lemma 6.13. *Let $f \in \Omega$ with $n := \operatorname{ar}(f)$. Let $k \in \mathbb{N}$.*

(i) *If $n > \deg(f)$, then f has a minor of degree $\deg(f)$ and arity $\deg(f) + 1$.*
(ii) *If $f \in \mathsf{X}_k$ and $\deg(f) > k$, then $n > \deg(f)$.*
(iii) *If $f \in \mathsf{X}_k$ and $n - 1 = \deg(f) > k$, then M_f contains all subsets of $[n]$ of cardinality $n - 1$.*
(iv) *If $f \in \mathsf{X}_k \setminus \mathsf{X}_{k-1}$, then f has a k-ary minor g such that $[k] \in M_g$ and $g \in \mathsf{D}_k \setminus \mathsf{X}_{k-1}$.*
(v) *If $f \in \mathsf{X}_k$ and there is an $S \in M_f$ with $\ell := |S| > k$, then f has an $(\ell+1)$-ary minor g such that M_g contains all subsets of $[\ell+1]$ of cardinality ℓ but $[\ell+1] \notin M_g$. Moreover, if $\ell > k + 1$, then M_g contains also a subset of cardinality $\ell - 1$.*
(vi) *If $\deg f = n$, then f has a minor of arity $n - 1$ and degree $n - 1$.*

Proof. (i) Let $m := \deg(f)$. There exists an $S \in M_f$ with $|S| = m$. Let us identify all arguments not in S, i.e., we form the minor f_σ with a minor formation map $\sigma\colon [n] \to [m+1]$ that maps S onto $[m]$ and every element of $[n] \setminus S$ to $m + 1$. Then f_σ has arity $m+1$. Clearly every monomial of f_σ has degree at most m, and $[m] \in M_{f_\sigma}$; hence $\deg(f_\sigma) = m$.

(ii) Clearly $n = \operatorname{ar}(f) \geq \deg(f)$. Assume that $n > k$, and suppose, to the contrary, that $n = \deg(f)$. But then $\operatorname{ch}([n-1], f) = 1$ and $|[n-1]| \geq k$, contradicting $f \in \mathsf{X}_k$.

(iii) Assume that $n - 1 = \deg(f) > k$. Then there exists an $S \in M_f$ with $|S| = n-1$. Let $A \subseteq S$ with $|A| = n-2$. Since $n-2 \geq k$ and $f \in \mathsf{X}_k$, there must be an even number of proper supersets of A in M_f. We already have $S \in M_f$, so there must be another one. In fact there is only one other possibility, namely $A \cup \{i\}$, where i is the unique element of $[n] \setminus S$. By letting A range over all $(n-2)$-element subsets of S, we conclude that M_f indeed contains all subsets of $[n]$ of cardinality $n - 1$.

(iv) Since $f \notin \mathsf{X}_{k-1}$, there exists a subset $A \subseteq [n]$ with $|A| = k - 1$ such that $\operatorname{ch}(A, f) = 1$. Let us identify all arguments not in A, i.e., we form the minor f_σ with a minor formation map $\sigma\colon [n] \to [k]$ that maps A onto $[k-1]$ and every element of $[n] \setminus A$ to k. Then f_σ has arity k. Since those subsets of $[n]$ whose image under σ equals $[k]$ are precisely all proper supersets of A, and since $\operatorname{ch}(A, f) = 1$, there are an odd number of sets $T \in M_f$ such that $\sigma(T) = [k]$. By Lemma 5.15, $[k] \in M_{f_\sigma}$. Then clearly $f_\sigma \in \mathsf{D}_k \setminus \mathsf{X}_{k-1}$.

(v) By part (ii), we must have $n > \deg(f) \geq \ell$. By identifying all arguments that are not in S, we obtain a minor g of f that has arity $\ell + 1$ and contains a monomial of degree ℓ. Since X_k is minor-closed, $g \in \mathsf{X}_k$, so by part (iii), $[\ell+1] \notin M_g$; hence $\deg(g) = \ell$. By part (iii), M_g contains all subsets of $[\ell+1]$ of cardinality ℓ. If $\ell > k+1$, then M_g must also contain a subset of cardinality $\ell - 1$. For, consider a subset $A \subseteq [\ell+1]$ with $|A| = \ell - 2$. Since $\ell - 2 \geq k$ and $g \in \mathsf{X}_k$, we have $\operatorname{ch}(A, g) = 0$, so there must be an even number of sets $S \in M_g$ with $A \subsetneq S$. There are exactly three such sets S of cardinality ℓ, namely $[\ell+1] \setminus \{i\}$ for each $i \in [\ell+1] \setminus A$; therefore there must also be a set of cardinality $\ell - 1$ in M_g.

(vi) If f has no monomial of degree $n - 1$, then for any $i, j \in [n]$ with $i < j$, the $(n-1)$-ary minor f_{ij} has degree $n - 1$. If f has exactly one monomial of degree $n - 1$, say $S \in M_f$, $|S| = n - 1$, then for any $i, j \in S$ with $i < j$, the minor f_{ij} has degree $n - 1$. If f has at least two monomials of degree $n - 1$, say $S, T \in M_f$,

$S \neq T$, $|S| = |T| = n - 1$, then for $\{i, j\} := S \triangle T$ with $i < j$, the minor f_{ij} has degree $n - 1$. $\square$

In what follows, we are going to make use of a family of special Boolean functions W_k that was inspired by the "unitrades" and the proof methods presented by Potapov [16, Section 4]. There is a minor difference in the definition, though. While Potapov's unitrade W_k is composed of all subsets of cardinality k, we nevertheless include all nonempty proper subsets of $[k + 1]$ in the set of monomials of W_k, as this will serve better our needs.

Definition 6.14. For $k \in \mathbb{N}$, let $W_k \colon \{0, 1\}^{k+1} \to \{0, 1\}$ be the function with $M_{W_k} = \{\, S \subseteq [k + 1] \mid 0 < |S| < k + 1 \,\}$. Equivalently, $W_k(\mathbf{a}) = 1$ if and only if $\mathbf{a} \notin \{(0, \dots, 0), (1, \dots, 1)\}$. For $n \geq k$ and $B \subseteq [n]$ with $|B| = k$, denote by W_k^B the minor $(W_k)_\sigma$ where $\sigma \colon [k] \to [n]$ is an injective map with range B (since W_k is totally symmetric, any such map σ produces the same minor). In other words, W_k^B is obtained from W_k by introducing $n - k$ fictitious arguments and then permuting arguments so that the essential arguments are the ones indexed by the elements of B. While the arity of W_k^B is not explicit in the notation, it will be clear from the context.

Lemma 6.15.

(i) *For any* $k \in \mathbb{N}$*, we have* $\deg(W_k) = k$, $\operatorname{par}(W_k) = 0$*, and* $\chi(W_k) = 0$*; hence* $W_k \in \mathsf{D}_k \cap \mathsf{X}_0 \cap \Omega_{00}$.

(ii) *For any* $k, \ell \in \mathbb{N}$ *with* $k \leq \ell$*,* W_k *is a minor of* W_ℓ.

Proof. (i) It is clear from the definition that $\deg(W_k) = k$. Since M_{W_k} comprises all subsets of $[k + 1]$ except $\emptyset$ and $[k + 1]$, we have $|M_{W_k} \setminus \{\emptyset\}| = 2^{k+1} - 2 = 2(2^k - 1)$, an even number; hence $\operatorname{par}(W_k) = 0$. As for the characteristic rank, for any $S \subseteq [k + 1]$, we have $\{\, A \in M_{W_k} \mid S \subsetneq A \,\} = \{\, A \subseteq [k + 1] \mid S \subsetneq A \subsetneq [k + 1] \,\}$. This set has $2^{k+1-|S|} - 2 = 2(2^{k-|S|} - 1)$ elements if $A \neq [k + 1]$ and no element if $A = [k + 1]$. Therefore $\operatorname{ch}(S, f) = 0$ for every $S \subseteq [k + 1]$, and we conclude that $\chi(f) = 0$. The constant term of W_k is clearly 0, and we conclude that $W_k \in \mathsf{D}_k \cap \mathsf{X}_0 \cap \Omega_= \cap \Omega_{0*} = \mathsf{D}_k \cap \mathsf{X}_0 \cap \Omega_{00}$.

(ii) By the transitivity of the minor relation, it suffices to show that W_k is a minor of W_{k+1} for any $k \in \mathbb{N}$. By identifying the $(k + 1)$-st and $(k + 2)$-nd arguments, i.e., by taking σ to be the identification map $\sigma_{k+1,k+2}$, we obtain, by Lemma 5.15,

$$M_{(W_{k+1})_\sigma} = \bigl\{\, S \subseteq [k + 1] \bigm| |\{\, T \in M_{W_{k+1}} \mid \sigma(T) = S \,\}| \equiv 1 \pmod{2} \,\bigr\} =: M.$$

We now determine which subsets of $[k + 1]$ belong to the set M on the right side of the above equality. Recall that $M_{W_{k+1}} = \{\, T \subseteq [k + 2] \mid 0 < |T| < k + 2 \,\}$. For any $S \subseteq [k]$, the only subset S' of $[k + 2]$ such that $\sigma(S) = \sigma(S')$ is S itself; hence $S \in M$ for all $\emptyset \neq S \subseteq [k]$. For any set of the form $S \cup \{k + 1\}$ with $S \subseteq [k]$, there are exactly three subsets S' of $[n + 2]$ such that $\sigma(S') = S \cup \{k+1\}$, namely the sets $S \cup \{k+1\}$, $S \cup \{k+2\}$, and $S \cup \{k+1, k+2\}$. If $S \neq [k]$, then all three sets belong to $M_{W_{k+1}}$. If $S = [k]$, then only the first two belong to $M_{W_{k+1}}$. Hence $S \cup \{k\} \in M$ for all $S \subsetneq [k]$. We conclude that $M = \{\, S \subseteq [k + 1] \mid 0 < |S| < k + 1 \,\} = M_{W_k}$, that is $(W_{k+1})_\sigma = W_k$. $\square$

Here is another functional construction that we will use in what follows.

Definition 6.16. For any function $f \colon \{0, 1\}^n \to \{0, 1\}$ and any $i \in [n]$, let $f'_i \colon \{0, 1\}^n \to \{0, 1\}$ be the function with $M_{f'_i} := \{\, S \setminus \{i\} \mid S \in M_f,\ i \in S \,\}$.

The effect of negating an argument in a function f can be expressed in a convenient way with the help of f'_i.

Lemma 6.17. *Let $f\colon \{0,1\}^n \to \{0,1\}$, $i \in [n]$, and let $g := f(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_n)$. Then $g = f + f'_i$.*

Proof. Given $f = \sum_{S \in M_f} x_S$, we have

$$g = f(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_n) = \sum_{\substack{S \in M_f \\ i \notin S}} x_S + \sum_{\substack{S \in M_f \\ i \in S}} (x_i + 1) x_{S \setminus \{i\}}$$
$$= \sum_{\substack{S \in M_f \\ i \notin S}} x_S + \sum_{\substack{S \in M_f \\ i \in S}} (x_S + x_{S \setminus \{i\}}) = \sum_{S \in M_f} x_S + \sum_{\substack{S \in M_f \\ i \in S}} x_{S \setminus \{i\}} = f + f'_i. \qquad \square$$

Lemma 6.18. *Let $f\colon \{0,1\}^n \to \{0,1\}$, and assume that f depends on the i-th argument.*

(i) *If $f \in \mathsf{X}_k$ for some $k > 0$, then $f'_i \in \mathsf{X}_{k-1}$.*
(ii) $\deg(f'_i) < \deg(f)$.

Proof. (i) Let $S \subseteq [n]$ with $|S| \geq k - 1$. If $i \in S$, then clearly $\mathrm{ch}(S, f'_i) = 0$ because no set in $M_{f'_i}$ contains the element i. If $i \notin S$, then a set A with $S \subsetneq A \subseteq [n]$ belongs to $M_{f'_i}$ if and only if $i \notin A$ and $A \cup \{i\} \in M_f$. Hence there is a one-to-one correspondence between the sets $\{\, A \in M_{f'_i} \mid S \subsetneq A \,\}$ and $\{\, B \in M_f \mid S \cup \{i\} \subseteq B \,\}$, so it follows that $\mathrm{ch}(S, f'_i) = \mathrm{ch}(S \cup \{i\}, f) = 0$, where the second equality holds because $|S \cup \{i\}| \geq k$ and $f \in \mathsf{X}_k$. We conclude that $f'_i \in \mathsf{X}_{k-1}$.

(ii) Obvious from the construction of f'_i. $\square$

Lemma 6.19. *For any $k \in \mathbb{N}$, every function in $\mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{00}$ is a sum of minors of W_k. Consequently, $\langle W_k \rangle_{\mathsf{L}_\mathrm{c}} = \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{00}$.*

Proof. We follow the proof technique of Potapov [16, Proposition 11]. Note that $\Omega_{00} \subseteq \Omega_{=}$, so every function in $\mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{00}$ is even. We proceed by induction on k. The claim is obvious for $k = 0$, since $\mathsf{D}_0 \cap \mathsf{X}_1 \cap \Omega_{00} = \mathsf{D}_0 \cap \Omega_{0*}$, and every constant 0 function (of any arity) can be obtained from W_0, the unary constant 0 function, by introducing fictitious arguments. The claim is also clear for $k = 1$, since $\mathsf{D}_1 \cap \mathsf{X}_1 \cap \Omega_{00} = \mathsf{D}_1 \cap \Omega_{=} \cap \Omega_{0*}$, and any even function of degree 1 with constant term 0 can be obtained by adding together suitable minors of $W_1 = x_1 + x_2$ obtained by introducing fictitious arguments and permuting arguments.

Assume now that the claim holds for $k = \ell$ for some $\ell \geq 1$. Every function in $\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$ of degree less than $\ell + 1$ is a sum of minors of W_ℓ by the induction hypothesis and is therefore a sum of minors of $W_{\ell+1}$ because $W_\ell \leq W_{\ell+1}$ by Lemma 6.15(ii). We only need to consider functions of degree exactly $\ell + 1$. We proceed by induction on the arity of functions. By Lemma 6.13(ii), for any $f \in \mathsf{X}_0$ with $\deg(f) = \ell + 1$, we must have $\mathrm{ar}(f) \geq \ell + 2$. Therefore, in order to establish the basis of induction, we need to consider an arbitrary function $f \in \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$ with $\mathrm{ar}(f) = \ell + 2$. By Lemma 6.13(iii), M_f contains all subsets of $[\ell + 2]$ of cardinality $\ell + 1$. Then $g := f + W_{\ell+1} = f + W_{\ell+1} + 0 \in \mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{00}$ because f, $W_{\ell+1}$, and 0 belong to $\mathsf{X}_1 \cap \Omega_{00}$, which is L_c-stable by Proposition 6.9, and $\deg(g) \leq \ell$ because all monomials of degree $\ell + 1$ are cancelled in the sum $f + W_{\ell+1}$. By the inductive hypothesis, g is a sum of minors of W_ℓ; hence $f = g + W_{\ell+1}$ is a sum of minors of $W_{\ell+1}$.

For the inductive step, assume that every m-ary function in $\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$ of degree $\ell + 1$ is a sum of minors of $W_{\ell+1}$. Let $f \in \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$ be $(m+1)$-ary and of degree $\ell + 1$. If f does not depend on the $(m+1)$-st argument, then f is obtained from an m-ary function $f^* \in \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$ by introducing a fictitious argument; then f^* is a sum of minors of $W_{\ell+1}$, and by introducing a fictitious argument to the summands we obtain f as a sum of minors of $W_{\ell+1}$. From now on, assume that f depends on the $(m+1)$-st argument. Let $g := f'_{m+1}$, and let c be the constant term (0 or 1) of g. By Lemma 6.18 we have $g \in \mathsf{D}_\ell \cap \mathsf{X}_0$; furthermore, $g + c \in \mathsf{D}_\ell \cap \mathsf{X}_0 \cap \Omega_{0*} = \mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{00}$. By the inductive hypothesis, $g + c$ is a sum of minors of W_ℓ, say $g + c = \sum_{i=1}^{p} W_{k_i}^{S_i}$, with $k_i \leq \ell$ for each i. Now let $h := \sum_{i=1}^{p} W_{k_i+1}^{S_i \cup \{m+1\}} + c^*$, where $c^* := 0$ if $c = 0$ and $c^* := W_1^{\{m,m+1\}}$ if $c = 1$, and let $f^* := f + h$. We have $f^* \in \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$ because f, h, and 0 belong to $\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$, which is L_c-stable. Moreover, f^* does not depend on the $(m+1)$-st argument because none of its monomials contains x_{m+1}. Let f^{**} be the m-ary function obtained from f^* by removing the fictitious $(m+1)$-st argument; then f^* and f^{**} are minors of each other. By the induction hypothesis, f^{**} is a sum of minors of $W_{\ell+1}$, and consequently so is f^* and hence also $f^* + h = f$.

As for the last claim about $\langle W_k \rangle_{\mathsf{L}_\mathsf{c}}$, since $0 = W_0$ is a minor of W_k, it follows that every sum of minors of W_k (not just every odd sum) is in $\langle W_k \rangle_{\mathsf{L}_\mathsf{c}}$. Therefore, by what we have shown above, $\mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{00} \subseteq \langle W_k \rangle_{\mathsf{L}_\mathsf{c}} \subseteq \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{00}$. $\square$

Lemma 6.20. *A Boolean function f belongs to X_k if and only if $f = g + h$ for some $g \in \mathsf{X}_0$ and $h \in \mathsf{D}_k$.*

Proof. "$\Leftarrow$": Clear because $\mathsf{X}_0 \subseteq \mathsf{X}_k$, $\mathsf{D}_k \subseteq \mathsf{X}_k$, and X_k is closed under sums by Lemma 6.8.

"$\Rightarrow$": We proceed by induction on k. For $k = 0$, the claim is obvious: if $f \in \mathsf{X}_0$, then $f = f + 0$, where $f \in \mathsf{X}_0$ and $0 \in \mathsf{D}_0$. For $k = 1$, this follows from Lemma 5.12: if $f \in \mathsf{X}_1 \cap \mathsf{X}_0$, then we are done by the above; if $f \in \mathsf{X}_1 \setminus \mathsf{X}_0$, then $f + x_1 \in \mathsf{X}_0$, so we have the decomposition $f = (f + x_1) + x_1$, where $f + x_1 \in \mathsf{X}_0$ and $x_1 \in \mathsf{D}_1$.

Assume now that the claim holds for $k = \ell$ for some $\ell \geq 0$. Let $f \in \mathsf{X}_{\ell+1}$. We proceed by induction on the degree of f. If $\deg(f) \leq \ell + 1$, then we clearly have $f = 0 + f$ with $0 \in \mathsf{X}_0$ and $f \in \mathsf{D}_{\ell+1}$. Assume that the claim holds for functions of degree at most $m \geq \ell + 1$. Consider now the case when $\deg(f) = m+1$. We proceed by induction on the arity n of f. By Lemma 6.13(ii), $n \geq m+2$. If $n = m+2$, then M_f contains all subsets of $[n]$ of cardinality $m + 1 = n - 1$ by Lemma 6.13(iii). Let $f^* := f + W_{m+1}$. Then $\deg(f^*) \leq m$, so by the induction hypothesis $f^* = g^* + h^*$ for some $g^* \in \mathsf{X}_0$ and $h^* \in \mathsf{D}_{\ell+1}$; therefore $f = (g^* + W_{m+1}) + h^*$, where $g^* + W_{m+1} \in \mathsf{X}_0$ by Lemma 6.8 and $h^* \in \mathsf{D}_{\ell+1}$. Assume that the claim holds for functions of arity p, and consider the case when $\mathrm{ar}(f) = p + 1$; we may assume that f depends on the $(p+1)$-st argument. By Lemma 6.18, we have $f'_{p+1} \in \mathsf{X}_\ell$, so by the induction hypothesis $f'_{p+1} = g^* + h^*$ for some $g^* \in \mathsf{X}_0$ and $h^* \in \mathsf{D}_\ell$; by changing the constant terms in g^* and h^* if necessary, we may assume that the constant term of g^* is 0. By Lemma 6.19, we can write g^* as $g^* = \sum_{i=1}^{s} W_{k_i}^{S_i}$. Let $g^+ := \sum_{i=1}^{s} W_{k_i+1}^{S_i \cup \{p+1\}}$, and let h^+ be the function with $M_{h^+} = \{\, S \cup \{p+1\} \mid S \in M_{h^*} \,\}$; then clearly $g^+ \in \mathsf{X}_0$ and $h^+ \in \mathsf{D}_{\ell+1}$. Let $\varphi := f + g^+ + h^+$. Clearly $\varphi \in \mathsf{X}_{\ell+1}$ and φ does not depend on the $(p+1)$-st argument, so by the induction hypothesis $\varphi = \gamma + \eta$ with $\gamma \in \mathsf{X}_0$ and $\eta \in \mathsf{D}_{\ell+1}$. Then $f = \varphi + g^+ + h^+ = (\gamma + g^+) + (\eta + h^+)$, where $\gamma + g^+ \in \mathsf{X}_0$ and $\eta + h^+ \in \mathsf{D}_{\ell+1}$, which gives us the desired decomposition. $\square$

Lemma 6.21. *For any $k \geq 2$, $\langle x_1 \dots x_k + x_1 \rangle_{\mathsf{L}_\mathsf{c}} = \mathsf{D}_k \cap \Omega_{00}$.*

Proof. Let $f := x_1 \dots x_k + x_1$. We have $f \in \mathsf{D}_k \cap \Omega_{00}$, so $\langle f \rangle_{\mathsf{L}_\mathsf{c}} \subseteq \mathsf{D}_k \cap \Omega_{00}$. By permuting arguments we get $g := x_1 x_{k+1} x_{k+2} \dots x_{2k-1} + x_1 \in \langle f \rangle_{\mathsf{L}_\mathsf{c}}$, and by identifying all arguments we get $0 \in \langle f \rangle_{\mathsf{L}_\mathsf{c}}$; hence also $h := f + g + 0 = x_1 \dots x_k + x_1 x_{k+1} x_{k+2} \dots x_{2k-1} \in \langle f \rangle_{\mathsf{L}_\mathsf{c}}$. Again by permuting the arguments of h we get $h' := x_1 x_{k+1} x_{k+2} \dots x_{2k-1} + x_{2k-1} x_{2k} \dots x_{3k-2} \in \langle f \rangle_{\mathsf{L}_\mathsf{c}}$; hence also $h'' := h + h' + 0 = x_1 \dots x_k + x_{2k-1} x_{2k} \dots x_{3k-2} \in \langle f \rangle_{\mathsf{L}_\mathsf{c}}$. It is clear that any even sum of monomials of degree at most k can be obtained by adding (an odd number of) minors of h''. Therefore $\mathsf{D}_k \cap \Omega_{00} = \mathsf{D}_k \cap \Omega_{=} \cap \Omega_{0*} \subseteq \langle h'' \rangle_{\mathsf{L}_\mathsf{c}} \subseteq \langle f \rangle_{\mathsf{L}_\mathsf{c}}$. $\square$

Proposition 6.22. *Let $a \in \{0,1\}$.*

(i) *Let $k \in \mathbb{N}_+$. For any $f \in (\mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{aa}) \setminus \mathsf{D}_{k-1}$, we have $\langle f \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{aa}$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $g \in (\mathsf{D}_k \cap \Omega_{aa}) \setminus \mathsf{X}_{k-1}$, we have $\langle g \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \Omega_{aa}$.*

(iii) *Let $i, j \in \mathbb{N}$ with $i > j \geq 2$. For any $f, g \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{aa}$ such that $f \notin \mathsf{D}_{i-1}$ and $g \notin \mathsf{X}_{j-1}$, we have $\langle f, g \rangle_{\mathsf{L_c}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{aa}$.*

Proof. It suffices to prove the statements for $a = 0$. The statements for $a = 1$ follow by Lemma 6.11 because $\overline{\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{11}$, $\overline{\mathsf{D}_{i-1}} = \mathsf{D}_{i-1}$, and $\overline{\mathsf{X}_{j-1}} = \mathsf{X}_{j-1}$. Note that $\Omega_{00} \subseteq \Omega_{=}$.

(i) We proceed by induction on k. For $k = 1$, let $f \in (\mathsf{D}_1 \cap \mathsf{X}_1 \cap \Omega_{00}) \setminus \mathsf{D}_0 = (\mathsf{D}_1 \cap \Omega_{=} \cap \Omega_{0*}) \setminus \mathsf{D}_0$. The function f is a sum of an even nonzero number of arguments, so by identification of arguments we get $0, x_1 + x_2 \in \langle f \rangle_{\mathsf{L_c}}$, and with these we can generate every even sum: $\mathsf{D}_1 \cap \mathsf{X}_1 \cap \Omega_{01} = \mathsf{D}_1 \cap \Omega_{=} \cap \Omega_{0*} \subseteq \langle f \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_1 \cap \mathsf{X}_1 \cap \Omega_{00}$.

Assume that the claim holds for $k = \ell$ for some $\ell \geq 1$. Let $f \in (\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}) \setminus \mathsf{D}_\ell$. Since $\mathsf{X}_1 \cap \Omega_{00} \subseteq \mathsf{X}_1 \cap \Omega_{=} = \mathsf{X}_0$, Lemma 6.13(v) implies that f has an $(\ell+2)$-ary minor φ such that $\deg(\varphi) = \ell + 1$ and M_φ contains all $(\ell+1)$-element subsets of $[\ell+2]$ and a subset S of cardinality ℓ. By identifying the two arguments not in S, we obtain a minor φ' of φ such that $\varphi' \in \mathsf{X}_0$, $\operatorname{ar}(\varphi') = \ell + 1$, and $\deg(\varphi') \geq \ell > 0$, so by Lemma 6.13(ii) we must have $\deg(\varphi') = \ell$. Since $\varphi' \in (\mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{00}) \setminus \mathsf{D}_{\ell-1}$, it holds that $\langle \varphi' \rangle_{\mathsf{L_c}} = \mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{00}$ by the induction hypothesis. All monomials of degree $\ell+1$ are cancelled in the sum $\varphi + W_{\ell+1}$, so we have $\varphi + W_{\ell+1} \in \mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{00} = \langle \varphi' \rangle_{\mathsf{L_c}} \subseteq \langle f \rangle_{\mathsf{L_c}}$. Since also $\varphi, 0 \in \langle f \rangle_{\mathsf{L_c}}$, we get $W_{\ell+1} = (\varphi + W_{\ell+1}) + \varphi + 0 \in \langle f \rangle_{\mathsf{L_c}}$. By Lemma 6.19, $\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00} = \langle W_{\ell+1} \rangle_{\mathsf{L_c}} \subseteq \langle f \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$.

(ii) We proceed by induction on k. For $k = 2$, let $g \in (\mathsf{D}_2 \cap \Omega_{00}) \setminus \mathsf{X}_1$. Since $\mathsf{D}_2 \subseteq \mathsf{X}_2$, g has a binary minor γ with $[2] \in M_\gamma$ by Lemma 6.13(iv). Since $\gamma \in \Omega_{00} \subseteq \Omega_{=}$, we have $\gamma \equiv x_1 x_2 + x_1$. It follows from Lemma 6.21 that $\mathsf{D}_2 \cap \Omega_{00} = \langle x_1 x_2 + x_1 \rangle_{\mathsf{L_c}} \subseteq \langle g \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_2 \cap \Omega_{00}$.

Assume that the claim holds for $k = \ell$ for some $\ell \geq 2$. Let $g \in (\mathsf{D}_{\ell+1} \cap \Omega_{00}) \setminus \mathsf{X}_\ell$. Since $\mathsf{D}_{\ell+1} \subseteq \mathsf{X}_{\ell+1}$, g has an $(\ell+1)$-ary minor γ with $[\ell+1] \in M_\gamma$ by Lemma 6.13(iv). By Lemma 6.13(vi), γ has an ℓ-ary minor $\gamma_{ij} \in (\mathsf{D}_\ell \cap \Omega_{00}) \setminus \mathsf{X}_{\ell-1}$. By the inductive hypothesis, $\mathsf{D}_\ell \cap \Omega_{00} = \langle \gamma_{ij} \rangle_{\mathsf{L_c}} \subseteq \langle g \rangle_{\mathsf{L_c}}$. The functions $\gamma' := \gamma + (x_1 \dots x_{\ell+1} + x_1)$ and 0 are members of $\mathsf{D}_\ell \cap \Omega_{00} \subseteq \langle g \rangle_{\mathsf{L_c}}$, so also $x_1 \dots x_{\ell+1} + x_1 = \gamma' + \gamma + 0 \in \langle g \rangle_{\mathsf{L_c}}$. By Lemma 6.21, $\mathsf{D}_{\ell+1} \cap \Omega_{00} = \langle x_1 \dots x_{\ell+1} + x_1 \rangle_{\mathsf{L_c}} \subseteq \langle g \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_{\ell+1} \cap \Omega_{00}$.

(iii) Let $f, g \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}$ such that $f \notin \mathsf{D}_{i-1}$ and $g \notin \mathsf{X}_{j-1}$. By Lemma 6.13(iv), g has a j-ary minor $g' \in \mathsf{D}_j \setminus \mathsf{X}_{j-1}$. Since $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}$ is minor-closed, we have $g' \in (\mathsf{D}_j \cap \Omega_{00}) \setminus \mathsf{X}_{j-1}$, and by part (ii), $\langle g' \rangle_{\mathsf{L_c}} = \mathsf{D}_j \cap \Omega_{00}$.

By Lemma 6.20, $f = f_1 + f_2$ for some $f_1 \in \mathsf{X}_0$ and $f_2 \in \mathsf{D}_j$. Since $\mathsf{X}_0 \subseteq \Omega_{=}$ and $f \in \Omega_{00} \subseteq \Omega_{=}$, we must also have $f_2 \in \Omega_{=}$. Since $f \in \Omega_{0*}$, it is clear that by changing the constant terms in f_1 and f_2 if necessary, we can assume that both f_1 and f_2 are in $\Omega_{=} \cap \Omega_{0*} = \Omega_{00}$. Thus $f_2 \in \mathsf{D}_j \cap \Omega_{00} = \langle g' \rangle_{\mathsf{L_c}} \subseteq \langle g \rangle_{\mathsf{L_c}}$, so $f_1 = f + f_2 + 0 \in \langle f, g \rangle_{\mathsf{L_c}}$. Since $f_1 \in (\mathsf{D}_i \cap \mathsf{X}_0 \cap \Omega_{0*}) \setminus \mathsf{D}_{i-1} = (\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{00}) \setminus \mathsf{D}_{i-1}$, we have $\langle f_1 \rangle_{\mathsf{L_c}} = \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{00}$ by part (i). It follows from Lemma 6.20 that

$$\begin{aligned}
\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00} &= \{\, \alpha + \beta \mid \alpha \in \mathsf{D}_i \cap \mathsf{X}_0 \cap \Omega_{00},\ \beta \in \mathsf{D}_j \cap \Omega_{00} \,\} \\
&= \{\, \alpha + \beta + 0 \mid \alpha \in \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{00},\ \beta \in \mathsf{D}_j \cap \Omega_{00} \,\} \\
&\subseteq \langle f_1, g' \rangle_{\mathsf{L_c}} \subseteq \langle f, g \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}.
\end{aligned}$$

□

Lemma 6.23. *For any $k \in \mathbb{N}_+$, $\langle x_1 \dots x_k \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \Omega_{01}$.*

Proof. It is clear that any monomial of degree at most k can be obtained as a minor of $x_1 \dots x_k$. Any function in $\mathsf{D}_k \cap \Omega_{01} = \mathsf{D}_k \cap \Omega_{\neq} \cap \Omega_{0*}$ is an odd sum of monomials of degree at most k. Therefore $\mathsf{D}_k \cap \Omega_{01} \subseteq \langle x_1 \dots x_k \rangle_{\mathsf{L}_\mathrm{c}} \subseteq \mathsf{D}_k \cap \Omega_{01}$. □

Proposition 6.24. *Let $a \in \{0, 1\}$.*

(i) *For any $f \in \mathsf{D}_1 \cap \Omega_{a\overline{a}}$, we have $\langle f \rangle_{\mathsf{L}_\mathrm{c}} = \mathsf{D}_1 \cap \Omega_{a\overline{a}}$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $f \in (\mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{a\overline{a}}) \setminus \mathsf{D}_{k-1}$, we have $\langle f \rangle_{\mathsf{L}_\mathrm{c}} = \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{a\overline{a}}$.*

(iii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $g \in (\mathsf{D}_k \cap \Omega_{a\overline{a}}) \setminus \mathsf{X}_{k-1}$, we have $\langle g \rangle_{\mathsf{L}_\mathrm{c}} = \mathsf{D}_k \cap \Omega_{a\overline{a}}$.*

(iv) *Let $i, j \in \mathbb{N}$ with $i > j \geq 2$. For any $f, g \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a\overline{a}}$ such that $f \notin \mathsf{D}_{i-1}$ and $g \notin \mathsf{X}_{j-1}$, we have $\langle f, g \rangle_{\mathsf{L}_\mathrm{c}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a\overline{a}}$.*

Proof. It suffices to prove the statements for $a = 0$. The statements for $a = 1$ follow by Lemma 6.11 because $\overline{\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{10}$, $\overline{\mathsf{D}_{i-1}} = \mathsf{D}_{i-1}$, and $\overline{\mathsf{X}_{j-1}} = \mathsf{X}_{j-1}$.

(i) If $f \in \mathsf{D}_1 \cap \Omega_{01}$, then by identifying all arguments we obtain $x_1 \in \langle f \rangle_{\mathsf{L}_\mathrm{c}}$. By Lemma 6.23, we have $\mathsf{D}_1 \cap \Omega_{01} = \mathsf{L}_\mathrm{c} = \langle x_1 \rangle_{\mathsf{L}_\mathrm{c}} \subseteq \langle f \rangle_{\mathsf{L}_\mathrm{c}} \subseteq \mathsf{D}_1 \cap \Omega_{01}$.

(ii) We show by induction on k that the claim holds for any $k \geq 1$ (not just for $k \geq 2$). The basis of the induction, the case when $k = 1$, is, in fact, statement (i) that we have already established; note that $(\mathsf{D}_1 \cap \mathsf{X}_1 \cap \Omega_{01}) \setminus \mathsf{D}_0 = \mathsf{D}_1 \cap \Omega_{01}$. For the induction step, assume that the claim holds for $k = \ell$ for some $\ell \geq 1$. Let $f \in (\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{01}) \setminus \mathsf{D}_\ell$. By Lemma 6.13(v), f has an $(\ell+2)$-ary minor $\varphi \in (\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{01}) \setminus \mathsf{D}_\ell$ such that M_φ contains all subsets of $[\ell+2]$ of cardinality $\ell+1$. If $\ell \geq 2$, then M_φ furthermore contains a subset of cardinality ℓ; then, again by Lemma 6.13(v), φ has an $(\ell+1)$-ary minor $\varphi' \in (\mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{01}) \setminus \mathsf{D}_{\ell-1}$, and by the inductive hypothesis, $\mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{01} = \langle \varphi' \rangle_{\mathsf{L}_\mathrm{c}} \subseteq \langle f \rangle_{\mathsf{L}_\mathrm{c}}$. If $\ell = 1$, then $\mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{01} = \mathsf{L}_\mathrm{c} = \langle x_1 \rangle_{\mathsf{L}_\mathrm{c}} \subseteq \langle f \rangle_{\mathsf{L}_\mathrm{c}}$ because x_1 is a minor of f (identify all arguments). In either case, let $\lambda := W_{\ell+1} + \varphi$. We have $W_{\ell+1} \in \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$ by Lemma 6.15 and $\varphi \in \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{01}$. Consequently $\lambda \in \mathsf{D}_\ell \cap \mathsf{X}_1 \cap \Omega_{01} \subseteq \langle f \rangle_{\mathsf{L}_\mathrm{c}}$ because all monomials of degree $\ell + 1$ are cancelled in the sum $W_{\ell+1} + \varphi$, X_1 is closed under sums by Lemma 6.8, and $\lambda(0, \dots, 0) = W_{\ell+1}(0, \dots, 0) + \varphi(0, \dots, 0) = 0 + 0 = 0$, $\lambda(1, \dots, 1) = W_{\ell+1}(1, \dots, 1) + \varphi(1, \dots, 1) = 0 + 1 = 1$.

Let now $h \in (\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{01}) \setminus \mathsf{D}_\ell$ be arbitrary. Then $h + x_1 \in \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{00}$, so by Lemma 6.19, $h + x_1 \in \langle W_{\ell+1} \rangle_{\mathsf{L}_\mathrm{c}}$, that is, $h + x_1 = \sum_{i=1}^{2m+1} W_{k_i}^{S_i}$ with $k_i \leq \ell + 1$. We can write $W_{k_i}^{S_i} = (W_{\ell+1})_{\sigma_i}$ for a suitable minor formation map σ_i. Consequently,

$$
\begin{aligned}
h &= \Big(\sum_{i=1}^{2m+1} W_{k_i}^{S_i} \Big) + x_1 = \Big(\sum_{i=1}^{2m+1} (W_{\ell+1})_{\sigma_i} \Big) + x_1 = \Big(\sum_{i=1}^{2m+1} (W_{\ell+1} + \varphi + \varphi)_{\sigma_i} \Big) + x_1 \\
&= \Big(\sum_{i=1}^{2m+1} \big(\underbrace{(W_{\ell+1} + \varphi)_{\sigma_i}}_{\in \langle f \rangle_{\mathsf{L}_\mathrm{c}}} + \underbrace{\varphi_{\sigma_i}}_{\in \langle f \rangle_{\mathsf{L}_\mathrm{c}}} \big) \Big) + \underbrace{x_1,}_{\in \langle f \rangle_{\mathsf{L}_\mathrm{c}}}
\end{aligned}
$$

where the last equality holds by Lemma 2.8. Since the last expression is an odd sum of elements of $\langle f \rangle_{\mathsf{L}_\mathrm{c}}$, it follows that $h \in \langle f \rangle_{\mathsf{L}_\mathrm{c}}$. We conclude that $\mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{01} \subseteq \langle f \rangle_{\mathsf{L}_\mathrm{c}} \subseteq \mathsf{D}_{\ell+1} \cap \mathsf{X}_1 \cap \Omega_{01}$.

(iii) We show by induction on k that the claim holds for any $k \geq 1$. The basis of the induction, the case when $k = 1$, is, in fact, statement (i) that we have already established, because $(\mathsf{D}_1 \cap \Omega_{01}) \setminus \mathsf{X}_0 = \mathsf{D}_1 \cap \Omega_{01}$. For the induction step, assume that the claim holds for $k = \ell$ for some $\ell \geq 1$. Let $g \in (\mathsf{D}_{\ell+1} \cap \Omega_{01}) \setminus \mathsf{X}_\ell$. By Lemma 6.13(iv), g has an $(\ell+1)$-ary minor γ such that $[\ell+1] \in M_\gamma$ and $\gamma \in (\mathsf{D}_{\ell+1} \cap \Omega_{01}) \setminus \mathsf{X}_\ell$. By Lemma 6.13(vi), γ has an ℓ-ary minor $\gamma_{ij} \in (\mathsf{D}_\ell \cap \Omega_{01}) \setminus \mathsf{X}_{\ell-1}$. By the inductive hypothesis, $\mathsf{D}_\ell \cap \Omega_{01} = \langle \gamma_{ij} \rangle_{\mathsf{L}_\mathrm{c}} \subseteq \langle g \rangle_{\mathsf{L}_\mathrm{c}}$. We have $\gamma' := \gamma + x_1 \dots x_{\ell+1} + x_1 \in$

$\mathsf{D}_\ell \cap \Omega_{01} \subseteq \langle g \rangle_{\mathsf{L_c}}$ and clearly $x_1 \in \langle g \rangle_{\mathsf{L_c}}$, so also $x_1 \dots x_{\ell+1} = \gamma' + \gamma + x_1 \in \langle g \rangle_{\mathsf{L_c}}$. By Lemma 6.23, we have $\mathsf{D}_{\ell+1} \cap \Omega_{01} = \langle x_1 \dots x_{\ell+1} \rangle_{\mathsf{L_c}} \subseteq \langle g \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_{\ell+1} \cap \Omega_{01}$.

(iv) Let $f, g \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}$ such that $f \notin \mathsf{D}_{i-1}$ and $g \notin \mathsf{X}_{j-1}$. By Lemma 6.13(iv), g has a j-ary minor $g' \in \mathsf{D}_j \setminus \mathsf{X}_{j-1}$. Since $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}$ is minor-closed, we have $g' \in (\mathsf{D}_j \cap \Omega_{01}) \setminus \mathsf{X}_{j-1}$, and by part (iii), $\langle g' \rangle_{\mathsf{L_c}} = \mathsf{D}_j \cap \Omega_{01}$.

By Lemma 6.20, $f = f_1 + f_2$ for some $f_1 \in \mathsf{X}_0$ and $f_2 \in \mathsf{D}_j$. Since $\mathsf{X}_0 \subseteq \Omega_=$ and $f \in \Omega_{\neq}$, we must also have $f_2 \in \Omega_{\neq}$. Since $f \in \Omega_{0*}$, it is clear that by changing the constant terms if necessary, we may assume that both f_1 and f_2 are in Ω_{0*}. Thus $f_2 \in \mathsf{D}_j \cap \Omega_{\neq} \cap \Omega_{0*} = \mathsf{D}_j \cap \Omega_{01} = \langle g' \rangle_{\mathsf{L_c}} \subseteq \langle g \rangle_{\mathsf{L_c}}$, so $f_1 + x_1 = f + f_2 + x_1 \in \langle f, g \rangle_{\mathsf{L_c}}$. Since $f_1 \in (\mathsf{D}_i \cap \mathsf{X}_0 \cap \Omega_{0*}) \setminus \mathsf{D}_{i-1} = (\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_= \cap \Omega_{0*}) \setminus \mathsf{D}_{i-1}$, we have $f_1 + x_1 \in (\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\neq} \cap \Omega_{0*}) \setminus \mathsf{D}_{i-1} = (\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{01}) \setminus \mathsf{D}_{i-1}$, so $\langle f_1 + x_1 \rangle_{\mathsf{L_c}} = \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{01}$ by part (ii).

Now, with the help of Lemma 6.20, we can see that for any $h \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}$, we have $h = h_1 + h_2$ for some $h_1 \in \mathsf{D}_i \cap \mathsf{X}_0 \cap \Omega_{0*} = \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_= \cap \Omega_{0*}$ and $h_2 \in \mathsf{D}_j \cap \Omega_{\neq} \cap \Omega_{0*} = \mathsf{D}_j \cap \Omega_{01}$, and hence $h_1 + x_1 \in \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{01} \subseteq \langle f, g \rangle_{\mathsf{L_c}}$ and $h_2 \in \langle g \rangle_{\mathsf{L_c}}$. Since $x_1 \in \langle f \rangle_{\mathsf{L_c}}$ as well, we have $h = (h_1 + x_1) + h_2 + x_1 \in \langle f, g \rangle_{\mathsf{L_c}}$. We conclude that $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01} \subseteq \langle f, g \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}$. ☐

Proposition 6.25. *Let $a, b \in \{0, 1\}$.*

(i) *For any $f_i \in (\mathsf{X}_1 \cap \Omega_{ab}) \setminus \mathsf{D}_i$ $(i \in \mathbb{N}_+)$, we have $\langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \rangle_{\mathsf{L_c}} = \mathsf{X}_1 \cap \Omega_{ab}$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $f_i \in (\mathsf{X}_k \cap \Omega_{ab}) \setminus \mathsf{D}_i$ $(i \in \mathbb{N}_+)$ and $g \in (\mathsf{X}_k \cap \Omega_{ab}) \setminus \mathsf{X}_{k-1}$, we have $\langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \cup \{g\} \rangle_{\mathsf{L_c}} = \mathsf{X}_k \cap \Omega_{ab}$.*

(iii) *For any $g_i \in (\Omega_{ab}) \setminus \mathsf{X}_i$ $(i \in \mathbb{N}_+)$, we have $\langle \{\, g_i \mid i \in \mathbb{N}_+ \,\} \rangle_{\mathsf{L_c}} = \Omega_{ab}$.*

Proof. (i) For $i \in \mathbb{N}_+$, let $f_i \in (\mathsf{X}_1 \cap \Omega_{ab}) \setminus \mathsf{D}_i$, and let $n_i := \deg(f_i)$; we have $n_i > i$. Then $f_i \in (\mathsf{D}_{n_i} \cap \mathsf{X}_1 \cap \Omega_{ab}) \setminus \mathsf{D}_{n_i - 1}$, so by Proposition 6.22(i) and Proposition 6.24(ii), $\langle f_i \rangle_{\mathsf{L_c}} = \mathsf{D}_{n_i} \cap \mathsf{X}_1 \cap \Omega_{ab}$. Therefore

$$\begin{aligned} \mathsf{X}_1 \cap \Omega_{ab} &= \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{ab}) \subseteq \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_{n_i} \cap \mathsf{X}_1 \cap \Omega_{ab}) \\ &= \bigcup_{i \in \mathbb{N}_+} \langle f_i \rangle_{\mathsf{L_c}} \subseteq \langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \rangle_{\mathsf{L_c}} \subseteq \mathsf{X}_1 \cap \Omega_{ab}. \end{aligned}$$

(ii) For $i \in \mathbb{N}_+$, let $f_i \in (\mathsf{X}_k \cap \Omega_{ab}) \setminus \mathsf{D}_i$, and let $g \in (\mathsf{X}_k \cap \Omega_{ab}) \setminus \mathsf{X}_{k-1}$, and let $n_i := \deg(f_i)$; we have $n_i > i$. By Lemma 6.13(iv), g has a k-ary minor γ of degree k such that $\gamma \in (\mathsf{D}_k \cap \Omega_{ab}) \setminus \mathsf{X}_{k-1}$. By Proposition 6.22(iii) and Proposition 6.24(iv), it holds for $i \geq k$ that $\langle f_i, g \rangle_{\mathsf{L_c}} = \mathsf{D}_{n_i} \cap \mathsf{X}_k \cap \Omega_{ab}$. Therefore

$$\begin{aligned} \mathsf{X}_k \cap \Omega_{ab} &= \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap \mathsf{X}_k \cap \Omega_{ab}) = \bigcup_{i \geq k} (\mathsf{D}_i \cap \mathsf{X}_k \cap \Omega_{ab}) \\ &\subseteq \bigcup_{i \geq k} (\mathsf{D}_{n_i} \cap \mathsf{X}_k \cap \Omega_{ab}) = \bigcup_{i \geq k} \langle f_i, g \rangle_{\mathsf{L_c}} \\ &\subseteq \langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \cup \{g\} \rangle_{\mathsf{L_c}} \subseteq \mathsf{X}_k \cap \Omega_{ab}. \end{aligned}$$

(iii) For $i \in \mathbb{N}_+$, let $g_i \in (\Omega_{ab}) \setminus \mathsf{X}_i$, and let $k_i := \chi(g_i)$. Then $g_i \in (\mathsf{X}_{k_i} \cap \Omega_{ab}) \setminus \mathsf{X}_{k_i - 1}$. By Lemma 6.13(iv), g_i has a k_i-ary minor γ_i of degree k_i such that $\gamma_i \in (\mathsf{D}_{k_i} \cap \Omega_{ab}) \setminus \mathsf{X}_{k_i - 1}$. By Proposition 6.22(ii) and Proposition 6.24(iii), $\langle \gamma_i \rangle_{\mathsf{L_c}} = \mathsf{D}_{k_i} \cap \Omega_{ab}$. Therefore

$$\begin{aligned} \Omega_{ab} &= \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap \Omega_{ab}) \subseteq \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_{k_i} \cap \Omega_{ab}) = \bigcup_{i \in \mathbb{N}_+} \langle \gamma_i \rangle_{\mathsf{L_c}} \\ &\subseteq \bigcup_{i \in \mathbb{N}_+} \langle g_i \rangle_{\mathsf{L_c}} \subseteq \langle \{\, g_i \mid i \in \mathbb{N}_+ \,\} \rangle_{\mathsf{L_c}} \subseteq \Omega_{ab}. \end{aligned}$$ ☐

Proposition 6.26. *Let $a \in \{0, 1\}$.*

(i) *Let $k \in \mathbb{N}_+$. For any $f, h, h' \in \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{a*}$ with $f \notin \mathsf{D}_{k-1}$, $h \notin \Omega_{*a}$, $h' \notin \Omega_{*\overline{a}}$, we have $\langle f, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{a*}$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $g, h, h' \in \mathsf{D}_k \cap \Omega_{a*}$ with $g \notin \mathsf{X}_{k-1}$, $h \notin \Omega_{*a}$, $h' \notin \Omega_{*\overline{a}}$, we have $\langle g, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \Omega_{a*}$.*

(iii) *Let $i, j \in \mathbb{N}$ with $i > j \geq 2$. For any $f, g, h, h' \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$ such that $f \notin \mathsf{D}_{i-1}$, $g \notin \mathsf{X}_{j-1}$, $h \notin \Omega_{*a}$, $h' \notin \Omega_{*\overline{a}}$, we have $\langle f, g, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$.*

Proof. It suffices to prove the statements for $a = 0$. The statements for $a = 1$ follow by Lemma 6.11 because $\overline{\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{1*}$, $\overline{\mathsf{D}_{i-1}} = \mathsf{D}_{i-1}$, $\overline{\mathsf{X}_{j-1}} = \mathsf{X}_{j-1}$, $\overline{\Omega_{*0}} = \Omega_{*1}$, and $\overline{\Omega_{*1}} = \Omega_{*0}$. We consider only statement (iii). The proofs of statements (i) and (ii) are analogous; we just need to omit the parts of the proof that deal with the function f or g, as the case may be, that does not appear in the statement.

Since $\{\Omega_{*0}, \Omega_{*1}\}$ is a partition of Ω, we have that $h \in \Omega_{*1}$ and $h' \in \Omega_{*0}$. By identifying all arguments, we get $x_1 \in \langle h \rangle_{\mathsf{L_c}}$ and $0 \in \langle h' \rangle_{\mathsf{L_c}}$, so we have $f + x_1 = f + x_1 + 0 \in \langle f, h, h' \rangle_{\mathsf{L_c}}$ and $g + x_1 = g + x_1 + 0 \in \langle g, h, h' \rangle_{\mathsf{L_c}}$. One of f and $f + x_1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}) \setminus \mathsf{D}_{i-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}) \setminus \mathsf{D}_{i-1}$, and, similarly, one of g and $g + x_1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}) \setminus \mathsf{X}_{j-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}) \setminus \mathsf{X}_{j-1}$. Propositions 6.22(iii) and 6.24(iv) imply that $\langle f, g, h, h' \rangle_{\mathsf{L_c}}$ contains a generating set for both $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}$ and $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}$. Therefore

$$\begin{aligned} \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*} &= (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}) \cup (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}) \\ &\subseteq \langle f, g, h, h' \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*}. \end{aligned}$$ □

Proposition 6.27. *Let $a \in \{0, 1\}$.*

(i) *Let $k \in \mathbb{N}_+$. For any $f, h, h' \in \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{*a}$ with $f \notin \mathsf{D}_{k-1}$, $h \notin \Omega_{a*}$, $h' \notin \Omega_{\overline{a}*}$, we have $\langle f, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{*a}$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $g, h, h' \in \mathsf{D}_k \cap \Omega_{*a}$ with $g \notin \mathsf{X}_{k-1}$, $h \notin \Omega_{a*}$, $h' \notin \Omega_{\overline{a}*}$, we have $\langle g, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \Omega_{*a}$.*

(iii) *Let $i, j \in \mathbb{N}$ with $i > j \geq 2$. For any $f, g, h, h' \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$ such that $f \notin \mathsf{D}_{i-1}$, $g \notin \mathsf{X}_{j-1}$, $h \notin \Omega_{a*}$, $h' \notin \Omega_{\overline{a}*}$, we have $\langle f, g, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$.*

Proof. It suffices to prove the statements for $a = 1$. The statements for $a = 0$ follow by Lemma 6.11 because $\overline{\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*1}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*0}$, $\overline{\mathsf{D}_{i-1}} = \mathsf{D}_{i-1}$, $\overline{\mathsf{X}_{j-1}} = \mathsf{X}_{j-1}$, $\overline{\Omega_{1*}} = \Omega_{0*}$, and $\overline{\Omega_{0*}} = \Omega_{1*}$. We consider only statement (iii). The proofs of statements (i) and (ii) are analogous; we just need to omit the parts of the proof that deal with the function f or g, as the case may be, that does not appear in the statement.

Since $\{\Omega_{0*}, \Omega_{1*}\}$ is a partition of Ω, we have that $h \in \Omega_{0*}$ and $h' \in \Omega_{1*}$. By identifying all arguments, we get $x_1 \in \langle h \rangle_{\mathsf{L_c}}$ and $1 \in \langle h' \rangle_{\mathsf{L_c}}$, so we have $f + x_1 + 1 \in \langle f, h, h' \rangle_{\mathsf{L_c}}$ and $g + x_1 + 1 \in \langle g, h, h' \rangle_{\mathsf{L_c}}$. One of f and $f + x_1 + 1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}) \setminus \mathsf{D}_{i-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{11}) \setminus \mathsf{D}_{i-1}$, and, similarly, one of g and $g + x_1 + 1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}) \setminus \mathsf{X}_{j-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{11}) \setminus \mathsf{X}_{j-1}$. Propositions 6.22(iii) and 6.24(iv) imply that $\langle f, g, h, h' \rangle_{\mathsf{L_c}}$ contains a generating set for both $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}$ and $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{11}$. Therefore

$$\begin{aligned} \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*1} &= (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{01}) \cup (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{11}) \\ &\subseteq \langle f, g, h, h' \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*1}. \end{aligned}$$ □

Proposition 6.28. *Let $\approx \in \{=, \neq\}$.*

(i) *Let $k \in \mathbb{N}_+$. For any $f, h, h' \in \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{\approx}$ with $f \notin \mathsf{D}_{k-1}$, $h \notin \Omega_{0*}$, $h' \notin \Omega_{1*}$, we have $\langle f, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \mathsf{X}_1 \cap \Omega_{\approx}$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $g, h, h' \in \mathsf{D}_k \cap \Omega_{\approx}$ with $g \notin \mathsf{X}_{k-1}$, $h \notin \Omega_{0*}$, $h' \notin \Omega_{1*}$, we have $\langle g, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \Omega_{\approx}$.*

(iii) *Let $i, j \in \mathbb{N}$ with $i > j \geq 2$. For any $f, g, h, h' \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}$ such that $f \notin \mathsf{D}_{i-1}$, $g \notin \mathsf{X}_{j-1}$, $h \notin \Omega_{0*}$, $h' \notin \Omega_{1*}$, we have $\langle f, g, h, h' \rangle_{\mathsf{L_c}} = \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}$.*

Proof. We consider only statement (iii). The proofs of statements (i) and (ii) are analogous; we just need to omit the parts of the proof that deal with the function f or g, as the case may be, that does not appear in the statement.

Since $\{\Omega_{0*}, \Omega_{1*}\}$ is a partition of Ω, we have that $h \in \Omega_{1*}$ and $h' \in \Omega_{0*}$. By identifying all arguments, we get $1 \in \langle h \rangle_{\mathsf{L_c}}$ and $0 \in \langle h' \rangle_{\mathsf{L_c}}$ if $\approx$ is $=$; or $x_1 + 1 \in \langle h \rangle_{\mathsf{L_c}}$ and $x_1 \in \langle h' \rangle_{\mathsf{L_c}}$ if $\approx$ is $\neq$. With the triple sum and these two minors of h and h' we are able to negate functions ($\varphi + 1 = \varphi + 1 + 0$ and $\varphi + 1 = \varphi + (x_1 + 1) + x_1$); hence $f + 1 \in \langle f, h, h' \rangle_{\mathsf{L_c}}$ and $g + 1 \in \langle g, h, h' \rangle_{\mathsf{L_c}}$. One of f and $f + 1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*} \cap \Omega_{\approx}) \setminus \mathsf{D}_{i-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{1*} \cap \Omega_{\approx}) \setminus \mathsf{D}_{i-1}$, and, similarly, one of g and $g+1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*} \cap \Omega_{\approx}) \setminus \mathsf{X}_{j-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{1*} \cap \Omega_{\approx}) \setminus \mathsf{X}_{j-1}$, Propositions 6.22(iii) and 6.24(iv) imply that $\langle f, g, h, h' \rangle_{\mathsf{L_c}}$ contains a generating set for both $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*} \cap \Omega_{\approx}$ and $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{1*} \cap \Omega_{\approx}$. Therefore

$$\begin{aligned} \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx} &= (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*} \cap \Omega_{\approx}) \cup (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{1*} \cap \Omega_{\approx}) \\ &\subseteq \langle f, g, h, h' \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}. \end{aligned}$$

□

Proposition 6.29. *Let $C \in \{\Omega_{0*}, \Omega_{1*}, \Omega_{*0}, \Omega_{*1}, \Omega_{=}, \Omega_{\neq}\}$, and let*

$$(K_1, K_2) := \begin{cases} (\Omega_{*0}, \Omega_{*1}) & \text{if } C \in \{\Omega_{0*}, \Omega_{1*}\}, \\ (\Omega_{0*}, \Omega_{1*}), & \text{if } C \in \{\Omega_{*0}, \Omega_{*1}, \Omega_{=}, \Omega_{\neq}\}. \end{cases}$$

(i) *For any $f_i \in (\mathsf{X}_1 \cap C) \setminus \mathsf{D}_i$ ($i \in \mathbb{N}_+$), $h_1 \in (\mathsf{X}_1 \cap C) \setminus K_1$, $h_2 \in (\mathsf{X}_1 \cap C) \setminus K_2$, we have $\langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \cup \{h_1, h_2\} \rangle_{\mathsf{L_c}} = \mathsf{X}_1 \cap C$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $f_i \in (\mathsf{X}_k \cap C) \setminus \mathsf{D}_i$ ($i \in \mathbb{N}_+$), $g \in (\mathsf{X}_k \cap C) \setminus \mathsf{X}_{k-1}$, $h_1 \in (\mathsf{X}_1 \cap C) \setminus K_1$, $h_2 \in (\mathsf{X}_1 \cap C) \setminus K_2$, we have $\langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \cup \{g, h_1, h_2\} \rangle_{\mathsf{L_c}} = \mathsf{X}_k \cap C$.*

(iii) *For any $g_i \in C \setminus \mathsf{X}_i$ ($i \in \mathbb{N}_+$), $h_1 \in C \setminus K_1$, $h_2 \in C \setminus K_2$, we have $\langle \{\, g_i \mid i \in \mathbb{N}_+ \,\} \cup \{h_1, h_2\} \rangle_{\mathsf{L_c}} = C$.*

Proof. (i) For $i \in \mathbb{N}_+$, let $f_i \in (\mathsf{X}_1 \cap C) \setminus \mathsf{D}_i$, $h_1 \in (\mathsf{X}_1 \cap C) \setminus K_1$, $h_2 \in (\mathsf{X}_1 \cap C) \setminus K_2$ and let $n_i := \deg(f_i)$; we have $n_i > i$. By identifying all arguments of h_1 and h_2, we get minors $\eta_1 \in (\mathsf{D}_1 \cap \mathsf{X}_1 \cap C) \setminus K_1$, $\eta_2 \in (\mathsf{D}_1 \cap \mathsf{X}_1 \cap C) \setminus K_2$. Since $f_i \in (\mathsf{D}_{n_i} \cap \mathsf{X}_1 \cap C) \setminus \mathsf{D}_{n_i - 1}$, it follows from Propositions 6.26(i), 6.27(i), and 6.28(i) that $\langle f_i, \eta_1, \eta_2 \rangle_{\mathsf{L_c}} = \mathsf{D}_{n_i} \cap \mathsf{X}_1 \cap C$ for any $i \in \mathbb{N}_+$. Therefore

$$\begin{aligned} \mathsf{X}_1 \cap C &= \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap \mathsf{X}_1 \cap C) \subseteq \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_{n_i} \cap \mathsf{X}_1 \cap C) \\ &= \bigcup_{i \in \mathbb{N}_+} \langle f_i, \eta_1, \eta_2 \rangle_{\mathsf{L_c}} \subseteq \langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \cup \{h_1, h_2\} \rangle_{\mathsf{L_c}} \subseteq \mathsf{X}_1 \cap C. \end{aligned}$$

(ii) For $i \in \mathbb{N}_+$, let $f_i \in (\mathsf{X}_k \cap C) \setminus \mathsf{D}_i$, $g \in (\mathsf{X}_k \cap C) \setminus \mathsf{X}_{k-1}$, $h_1 \in (\mathsf{X}_1 \cap C) \setminus K_1$, $h_2 \in (\mathsf{X}_1 \cap C) \setminus K_2$, and let $n_i := \deg(f_i)$; we have $n_i > i$. By Lemma 6.13(iv), g has a k-ary minor γ of degree k such that $\gamma \in (\mathsf{D}_k \cap \mathsf{X}_k \cap C) \setminus \mathsf{X}_{k-1}$. By identifying all arguments of h_1 and h_2, we get minors $\eta_1 \in (\mathsf{D}_1 \cap \mathsf{X}_k \cap C) \setminus K_1$, $\eta_2 \in (\mathsf{D}_1 \cap \mathsf{X}_k \cap C) \setminus K_2$. By Propositions 6.26(iii), 6.27(iii), and 6.28(iii) it holds that $\langle f_i, g, \eta_1, \eta_2 \rangle_{\mathsf{L_c}} = \mathsf{D}_{n_i} \cap \mathsf{X}_k \cap C$ whenever $n_i \geq k$ (this certainly holds whenever $i \geq k$). Therefore

$$\begin{aligned} \mathsf{X}_k \cap C &= \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap \mathsf{X}_k \cap C) = \bigcup_{i \geq k} (\mathsf{D}_i \cap \mathsf{X}_k \cap C) \\ &\subseteq \bigcup_{i \geq k} (\mathsf{D}_{n_i} \cap \mathsf{X}_k \cap C) = \bigcup_{i \geq k} \langle f_i, g, \eta_1, \eta_2 \rangle_{\mathsf{L_c}} \\ &\subseteq \langle \{\, f_i \mid i \in \mathbb{N}_+ \,\} \cup \{g, h_1, h_2\} \rangle_{\mathsf{L_c}} \subseteq \mathsf{X}_k \cap C. \end{aligned}$$

(iii) For $i \in \mathbb{N}_+$, let $g_i \in C \setminus \mathsf{X}_i$, $h_1 \in C \setminus K_1$, $h_2 \in C \setminus K_2$, and let $k_i := \chi(g_i)$. Then $g_i \in (\mathsf{X}_{k_i} \cap C) \setminus \mathsf{X}_{k_i-1}$. By Lemma 6.13(iv), g_i has a k_i-ary minor γ_i of degree k_i such that $\gamma_i \in (\mathsf{D}_{k_i} \cap C) \setminus \mathsf{X}_{k_i-1}$. By identifying all arguments of h_1 and h_2, we get minors $\eta_1 \in (\mathsf{D}_1 \cap C) \setminus K_1$, $\eta_2 \in (\mathsf{D}_1 \cap C) \setminus K_2$. By Propositions 6.26(ii), 6.27(ii), and 6.28(ii) it holds that $\langle \gamma_i, \eta_1, \eta_2 \rangle_{\mathsf{L_c}} = \mathsf{D}_{k_i} \cap C$ for any $i \in \mathbb{N}_+$. Therefore

$$\begin{aligned} C = \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap C) &\subseteq \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_{k_i} \cap C) = \bigcup_{i \in \mathbb{N}_+} \langle \gamma_i, \eta_1, \eta_2 \rangle_{\mathsf{L_c}} \\ &\subseteq \bigcup_{i \in \mathbb{N}_+} \langle g_i, h_1, h_2 \rangle_{\mathsf{L_c}} \subseteq \langle \{\, g_i \mid i \in \mathbb{N}_+ \,\} \cup \{h_1, h_2\} \rangle_{\mathsf{L_c}} \subseteq C. \qquad \square \end{aligned}$$

Lemma 6.30. *For any $h_1 \in \Omega_{0*}$, $h_2 \in \Omega_{1*}$, $h_3 \in \Omega_{*0}$, $h_4 \in \Omega_{*1}$, $h_5 \in \Omega_{=}$, $h_6 \in \Omega_{\neq}$, we have $\mathsf{D}_1 \subseteq \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$.*

Proof. By identifying all arguments, we see that

$$\begin{array}{lcccccl} & \text{either} & 0 & \text{or} & x_1 & \text{is in} & \langle h_1 \rangle_{\mathsf{L_c}}, \\ & \text{either} & 1 & \text{or} & x_1+1 & \text{is in} & \langle h_2 \rangle_{\mathsf{L_c}}, \\ & \text{either} & 0 & \text{or} & x_1+1 & \text{is in} & \langle h_3 \rangle_{\mathsf{L_c}}, \\ (3) & \text{either} & 1 & \text{or} & x_1 & \text{is in} & \langle h_4 \rangle_{\mathsf{L_c}}, \\ & \text{either} & 0 & \text{or} & 1 & \text{is in} & \langle h_5 \rangle_{\mathsf{L_c}}, \\ & \text{either} & x_1 & \text{or} & x_1+1 & \text{is in} & \langle h_6 \rangle_{\mathsf{L_c}}. \end{array}$$

Let $G := \{0, 1, x_1, x_1+1\}$. Clearly $\langle G \rangle_{\mathsf{L_c}} = \mathsf{D}_1$. Any three-element subset of G also generates D_1 because each element of G is the sum of the other three elements. Any choice of functions from the six pairs in (3) includes at least three different elements of G, so we conclude that $\mathsf{D}_1 \subseteq \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$. $\square$

Proposition 6.31.

(i) *Let $k \in \mathbb{N}_+$. For any $f, h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{D}_k \cap \mathsf{X}_1$ with $f \notin \mathsf{D}_{k-1}$, $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, we have $\langle f, g_1, g_2, g_3, g_4, g_5, g_6 \rangle_{\mathsf{L_c}} = \mathsf{D}_k \cap \mathsf{X}_1$.*

(ii) *Let $k \in \mathbb{N}_+$ with $k \geq 2$. For any $g, h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{D}_k$ with $g \notin \mathsf{X}_{k-1}$, $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, we have $\langle g, h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}} = \mathsf{D}_k$.*

(iii) *Let $i, j \in \mathbb{N}$ with $i > j \geq 2$. For any $f, g, h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{D}_i \cap \mathsf{X}_j$ with $f \notin \mathsf{D}_{i-1}$, $g \notin \mathsf{X}_{j-1}$, $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, we have $\langle f, g, h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}} = \mathsf{D}_i \cap \mathsf{X}_j$.*

Proof. We consider only statement (iii). The proofs of statements (i) and (ii) are analogous; we just need to omit the parts of the proof that deal with the function f or g, as the case may be, that does not appear in the statement.

Since $\{\Omega_{=}, \Omega_{\neq}\}$, $\{\Omega_{0*}, \Omega_{1*}\}$, and $\{\Omega_{*0}, \Omega_{*1}\}$ are partitions of Ω, we have that $h_1 \in \Omega_{1*}$, $h_2 \in \Omega_{0*}$, $h_3 \in \Omega_{*1}$, $h_4 \in \Omega_{*0}$, $h_5 \in \Omega_{\neq}$, and $h_6 \in \Omega_{=}$. By Lemma 6.30, we have $\mathsf{D}_1 \subseteq \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$. Hence $f + x_1 + 1 \in \langle f, h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$ and $g + x_1 + 1 \in \langle g, h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$. One of f and $f + x_1 + 1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}) \setminus \mathsf{D}_{i-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}) \setminus \mathsf{D}_{i-1}$, and, similarly, one of g and $g + x_1 + 1$ belongs to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}) \setminus \mathsf{X}_{j-1}$ and the other to $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}) \setminus \mathsf{X}_{j-1}$. Proposition 6.28(iii) implies that $\langle f, g, h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$ contains a generating set for both $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}$ and $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$. Therefore

$$\begin{aligned} \mathsf{D}_i \cap \mathsf{X}_j &= (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}) \cup (\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}) \\ &\subseteq \langle f, g, h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}} \subseteq \mathsf{D}_i \cap \mathsf{X}_j. \qquad \square \end{aligned}$$

Proposition 6.32.

(i) *For any* $f_i \in \mathsf{X}_1 \setminus \mathsf{D}_i$ $(i \in \mathbb{N}_+)$ *and* $h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{X}_1$ *such that* $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, *we have* $\langle \{ f_i \mid i \in \mathbb{N}_+ \} \cup \{h_1, h_2, h_3, h_4, h_5, h_6\} \rangle_{\mathsf{L_c}} = \mathsf{X}_1$.

(ii) *Let* $k \in \mathbb{N}_+$ *with* $k \geq 2$. *For any* $f_i \in \mathsf{X}_k \setminus \mathsf{D}_i$ $(i \in \mathbb{N}_+)$, $g \in \mathsf{X}_k \setminus \mathsf{X}_{k-1}$, *and* $h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{X}_k$ *such that* $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, *we have* $\langle \{ f_i \mid i \in \mathbb{N}_+ \} \cup \{g, h_1, h_2, h_3, h_4, h_5, h_6\} \rangle_{\mathsf{L_c}} = \mathsf{X}_k$.

(iii) *For any* $g_i \in \Omega \setminus \mathsf{X}_i$ $(i \in \mathbb{N}_+)$ *and* $h_1, h_2, h_3, h_4, h_5, h_6 \in \Omega$ *such that* $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, *we have* $\langle \{ g_i \mid i \in \mathbb{N}_+ \} \cup \{h_1, h_2, h_3, h_4, h_5, h_6\} \rangle_{\mathsf{L_c}} = \Omega$.

Proof. Observe first that $0, 1, x_1 \in \mathsf{D}_1 \subseteq \mathsf{X}_k$ for any $k \in \mathbb{N}_+$ and $1 \notin \Omega_{0*}$, $0 \notin \Omega_{1*}$, $1 \notin \Omega_{*0}$, $0 \notin \Omega_{*1}$, $x_1 \notin \Omega_{=}$, $0 \notin \Omega_{\neq}$.

(i) For $i \in \mathbb{N}_+$, let $f_i \in \mathsf{X}_1 \setminus \mathsf{D}_i$ and $h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{X}_1$ be such that $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, and let $n_i := \deg(f_i)$; we have $n_i > i$. Since $f_i \in (\mathsf{D}_{n_i} \cap \mathsf{X}_1) \setminus \mathsf{D}_{n_i - 1}$, Proposition 6.31(i) implies $\langle f_i, 0, 1, x_1 \rangle_{\mathsf{L_c}} = \mathsf{D}_{n_i} \cap \mathsf{X}_1$ for any $i \in \mathbb{N}_+$. We have $\{0, 1, x_1\} \subseteq \mathsf{D}_1 \subseteq \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$ by Lemma 6.30. Therefore

$$
\begin{aligned}
\mathsf{X}_1 = \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap \mathsf{X}_1) \subseteq \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_{n_i} \cap \mathsf{X}_1) = \bigcup_{i \in \mathbb{N}_+} \langle f_i, 0, 1, x_1 \rangle_{\mathsf{L_c}} \\
\subseteq \langle \{ f_i \mid i \in \mathbb{N}_+ \} \cup \{h_1, h_2, h_3, h_4, h_5, h_6\} \rangle_{\mathsf{L_c}} \subseteq \mathsf{X}_1.
\end{aligned}
$$

(ii) For $i \in \mathbb{N}_+$, let $f_i \in \mathsf{X}_k \setminus \mathsf{D}_i$, $g \in \mathsf{X}_k \setminus \mathsf{X}_{k-1}$, and $h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{X}_k$ such that $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, and let $n_i := \deg(f_i)$; we have $n_i > i$. By Lemma 6.13(iv), g has a k-ary minor γ of degree k such that $\gamma \in \mathsf{X}_k \setminus \mathsf{X}_{k-1}$; hence $\gamma \in \mathsf{D}_k \setminus \mathsf{X}_{k-1}$. By Proposition 6.31(iii), it holds that $\langle f_i, \gamma, 0, 1, x_1 \rangle_{\mathsf{L_c}} = \mathsf{D}_{n_i} \cap \mathsf{X}_k$ whenever $n_i \geq k$. We have $\{0, 1, x_1\} \subseteq \mathsf{D}_1 \subseteq \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$ by Lemma 6.30. Therefore

$$
\begin{aligned}
\mathsf{X}_k = \bigcup_{i \in \mathbb{N}_+} (\mathsf{D}_i \cap \mathsf{X}_k) = \bigcup_{i \geq k} (\mathsf{D}_i \cap \mathsf{X}_k) \subseteq \bigcup_{i \geq k} (\mathsf{D}_{n_i} \cap \mathsf{X}_k) = \bigcup_{i \geq k} \langle f_i, \gamma, 0, 1, x_1 \rangle_{\mathsf{L_c}} \\
\subseteq \langle \{ f_i \mid i \in \mathbb{N}_+ \} \cup \{g, h_1, h_2, h_3, h_4, h_5, h_6\} \rangle_{\mathsf{L_c}} \subseteq \mathsf{X}_k.
\end{aligned}
$$

(iii) For $i \in \mathbb{N}_+$, let $g_i \in \Omega \setminus \mathsf{X}_i$, and $h_1, h_2, h_3, h_4, h_5, h_6 \in \mathsf{X}_k$ such that $h_1 \notin \Omega_{0*}$, $h_2 \notin \Omega_{1*}$, $h_3 \notin \Omega_{*0}$, $h_4 \notin \Omega_{*1}$, $h_5 \notin \Omega_{=}$, $h_6 \notin \Omega_{\neq}$, and let $k_i := \chi(g_i)$. Then $g_i \in \mathsf{X}_{k_i} \setminus \mathsf{X}_{k_i - 1}$. By Lemma 6.13(iv), g_i has a k_i-ary minor γ_i of degree k_i such that $\gamma_i \in \mathsf{D}_{k_i} \setminus \mathsf{X}_{k_i - 1}$. By Proposition 6.31(ii), it holds that $\langle \gamma_i, 0, 1, x_1 \rangle_{\mathsf{L_c}} = \mathsf{D}_{k_i}$ for any $i \in \mathbb{N}_+$. We have $\{0, 1, x_1\} \subseteq \mathsf{D}_1 \subseteq \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle_{\mathsf{L_c}}$ by Lemma 6.30. Therefore

$$
\begin{aligned}
\Omega = \bigcup_{i \in \mathbb{N}_+} \mathsf{D}_i \subseteq \bigcup_{i \in \mathbb{N}_+} \mathsf{D}_{k_i} = \bigcup_{i \in \mathbb{N}_+} \langle \gamma_i, 0, 1, x_1 \rangle_{\mathsf{L_c}} \\
\subseteq \langle \{ g_i \mid i \in \mathbb{N}_+ \} \cup \{h_1, h_2, h_3, h_4, h_5, h_6\} \rangle_{\mathsf{L_c}} \subseteq \Omega. \qquad \square
\end{aligned}
$$

Proof of Theorem 6.1. By Lemma 6.2(iii), $\mathsf{L_c}$-stability is equivalent to $(\mathsf{I_c}, \mathsf{L_c})$-stability. The given classes are $\mathsf{L_c}$-stable by Proposition 6.9. The fact that there are no further $\mathsf{L_c}$-stable classes distinct from these follows from Propositions 6.12, 6.22, 6.24, 6.25, 6.26, 6.27, 6.28, 6.29, 6.31, 6.32, in which we have shown that any set of Boolean functions generates one of the classes listed in the statement – more precisely, that for each class C and for any set $F \subseteq C$ that is not included in any proper subclass of C it holds that $\langle F \rangle_{\mathsf{L_c}} = C$. $\square$

7. (C_1, C_2)-STABLE CLASSES FOR $\mathsf{L_c} \subseteq C_2$

Theorem 6.1 allows us to describe also all (C_1, C_2)-stable classes of Boolean functions for clones C_1 and C_2 such that C_1 is arbitrary and $\mathsf{L_c} \subseteq C_2$. Namely, by

Lemma 6.2, L_{c}-stability is equivalent to $(\mathsf{I}_{\mathrm{c}}, \mathsf{L}_{\mathrm{c}})$-stability. Since (C_1, C_2)-stability implies $(\mathsf{I}_{\mathrm{c}}, \mathsf{L}_{\mathrm{c}})$-stability whenever $\mathsf{L}_{\mathrm{c}} \subseteq C_2$, it suffices to search for (C_1, C_2)-stable classes among the L_{c}-stable ones. To this end, we determine, for each $(\mathsf{I}_{\mathrm{c}}, \mathsf{L}_{\mathrm{c}})$-stable class K, the clones C_1 and C_2 for which it holds that $KC_1 \subseteq K$ and $C_2K \subseteq K$. The results are summarized in the following theorem which refers to Table 2.

Theorem 7.1. *For each L_{c}-stable class K, as determined in Theorem 6.1, there exist clones C_1^K and C_2^K, as prescribed in Table 2, such that for every clone C, it holds that $KC \subseteq K$ if and only if $C \subseteq C_1^K$, and $CK \subseteq K$ if and only if $C \subseteq C_2^K$.*

The proof of Theorem 7.1 will be developed in the remainder of this section. The following two lemmata will be useful. The first one (Lemma 7.2) provides sufficient conditions for right and left stability for classes that are intersections of classes for which we already know sufficient conditions for right and left stability. The second one (Lemma 7.3) provides necessary conditions. These will be applied in the subsequent propositions in which necessary and sufficient stability conditions are established for each L_{c}-stable class.

Lemma 7.2. *Let $K_1, K_2, C_1, C_2 \subseteq \Omega$. Then the following statements hold.*

(i) *Assume that $K_1C \subseteq K_1$ whenever $C \subseteq C_1$ and $K_2C \subseteq K_2$ whenever $C \subseteq C_2$. Then $(K_1 \cap K_2)C \subseteq K_1 \cap K_2$ whenever $C \subseteq C_1 \cap C_2$.*

(ii) *Assume that $CK_1 \subseteq K_1$ whenever $C \subseteq C_1$ and $CK_2 \subseteq K_2$ whenever $C \subseteq C_2$. Then $C(K_1 \cap K_2) \subseteq K_1 \cap K_2$ whenever $C \subseteq C_1 \cap C_2$.*

Proof. (i) If $C \subseteq C_1 \cap C_2$, then $(K_1 \cap K_2)C \subseteq K_1C_1 \subseteq K_1$ and $(K_1 \cap K_2)C \subseteq K_2C_2 \subseteq K_2$ by the monotonicity of function class composition and the stability of K_1 and K_2 under right composition with C_1 and C_2, respectively. Therefore $(K_1 \cap K_2)C \subseteq K_1 \cap K_2$.

(ii) The proof is analogous to that of part (i). $\square$

Lemma 7.3. *Let $a, b \in \{0, 1\}$, $\approx \in \{=, \neq\}$, $i, j \in \mathbb{N}_+$ with $i \geq j \geq 1$.*

(i) *For any $\emptyset \neq K \subseteq \Omega$, the following statements hold.*
  (a) $\mathsf{I}_{\overline{a}}K \nsubseteq \Omega_{a*} \cup \Omega_{*a}$.
  (b) $\mathsf{I}_aK \nsubseteq \Omega_{\neq}$.
  (c) *If $a \neq b$, then* $\mathsf{I}_0K \nsubseteq \Omega_{ab}$, $\mathsf{I}_1K \nsubseteq \Omega_{ab}$.

(ii) *For $K := \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$, the following statements hold.*
  (d) $\mathsf{I}^*K \nsubseteq \Omega_{a*} \cup \Omega_{*b}$.
  (e) $\Lambda_{\mathrm{c}}K \nsubseteq \mathsf{D}_i$, $\mathsf{V}_{\mathrm{c}}K \nsubseteq \mathsf{D}_i$. *If $j \geq 2$ or $a \neq b$, then* $\Lambda_{\mathrm{c}}K \nsubseteq \mathsf{X}_j$, $\mathsf{V}_{\mathrm{c}}K \nsubseteq \mathsf{X}_j$.
  (f) $\mathsf{SM}K \nsubseteq \mathsf{D}_i$. *If $j \geq 2$, then* $\mathsf{SM}K \nsubseteq \mathsf{X}_j$.
  (g) $K\mathsf{I}_0 \nsubseteq \Omega_{*b}$, $K\mathsf{I}_1 \nsubseteq \Omega_{a*}$, $K\mathsf{I}^* \nsubseteq \Omega_{a*} \cup \Omega_{*b}$.
  (h) *If $i > j$, then* $K\mathsf{I}_0 \nsubseteq \mathsf{X}_j$, $K\mathsf{I}_1 \nsubseteq \mathsf{X}_j$.
  (i) $K\Lambda_{\mathrm{c}} \nsubseteq \mathsf{D}_i \cup \mathsf{X}_j$, $K\mathsf{V}_{\mathrm{c}} \nsubseteq \mathsf{D}_i \cup \mathsf{X}_j$,
  (j) $K\mathsf{SM} \nsubseteq \mathsf{D}_i$. *If $j \geq 2$, then* $K\mathsf{SM} \nsubseteq \mathsf{X}_j$.

(iii) (k) $\mathsf{SM}(\mathsf{X}_1 \cap \Omega_{a*}) \nsubseteq \mathsf{X}_1$, $\mathsf{SM}(\mathsf{X}_1 \cap \Omega_{*a}) \nsubseteq \mathsf{X}_1$.

(iv) *For $K := \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}$, the following statements hold.*
  (l) $K\mathsf{I}_0 \nsubseteq \Omega_{\approx}$, $K\mathsf{I}_1 \nsubseteq \Omega_{\approx}$.
  (m) *If $j \geq 2$, then* $K\mathsf{I}^* \nsubseteq \Omega_{\approx}$.
  (n) *If $\approx = \neq$, then* $\Lambda_{\mathrm{c}}K \nsubseteq \Omega_{\approx}$, $\mathsf{V}_{\mathrm{c}}K \nsubseteq \Omega_{\approx}$,

Proof. Throughout the proof, we will use Lemmata 3.3 and 3.2 together with the fact that $\mathsf{I}_0 = \langle 0 \rangle$, $\mathsf{I}_1 = \langle 1 \rangle$, $\mathsf{I}^* = \langle x_1 + 1 \rangle$, $\Lambda_{\mathrm{c}} = \langle \wedge \rangle$, $\mathsf{V}_{\mathrm{c}} = \langle \vee \rangle$, $\mathsf{SM} = \langle \mu \rangle$.

(i) (a) For any $\varphi \in \Omega$, we have $\overline{a}(\varphi) = \overline{a} \notin \Omega_{a*} \cup \Omega_{*a}$. Therefore $\mathsf{I}_{\overline{a}}K \nsubseteq \Omega_{a*} \cup \Omega_{*a}$.

(b) For any $\varphi \in \Omega$, we have $a(\varphi) = a \notin \Omega_{\neq}$. Therefore $\mathsf{I}_aK \nsubseteq \Omega_{\neq}$.

(c) If $a \neq b$, then, by (a), we have $\mathsf{I}_0K \nsubseteq \Omega_{1*} \cup \Omega_{*1}$ and $\mathsf{I}_1K \nsubseteq \Omega_{0*} \cup \Omega_{*0}$. Since $\Omega_{*a}b$ is a subset of both $\Omega_{0*} \cup \Omega_{*0}$ and $\Omega_{1*} \cup \Omega_{*1}$, it follows that $\mathsf{I}_iK \nsubseteq \Omega_{ab}$ for $i \in \{0, 1\}$.

K		$KC \subseteq K$ if and only if $C \subseteq \dots$	$CK \subseteq K$ if and only if $C \subseteq \dots$	result
Ω		Ω	Ω	Proposition 7.4
Ω_{a*}		T_0	T_a	Proposition 7.5
Ω_{*a}		T_1	T_a	Proposition 7.5
$\Omega_{=}$		T_c	Ω	Proposition 7.7
$\Omega_{\neq}$		T_c	S	Proposition 7.7
Ω_{ab}		T_c	$\mathsf{T}_a \cap \mathsf{T}_b$	Proposition 7.8
X_k	$k \geq 2$	LS	L	Proposition 7.10
	$k = 1$	S	L	
$\mathsf{X}_k \cap \Omega_{a*}$	$k \geq 2$	L_c	L_a	Proposition 7.11
	$k = 1$	S_c	L_a	
$\mathsf{X}_k \cap \Omega_{*a}$	$k \geq 2$	L_c	L_a	Proposition 7.12
	$k = 1$	S_c	L_a	
$\mathsf{X}_k \cap \Omega_{=}$	$k \geq 2$	L_c	L	Proposition 7.13
	$k = 1$	S	Ω	
$\mathsf{X}_k \cap \Omega_{\neq}$	$k \geq 2$	L_c	LS	Proposition 7.14
	$k = 1$	S	S	
$\mathsf{X}_k \cap \Omega_{ab}$	$k \geq 2$	L_c	$\mathsf{L}_a \cap \mathsf{L}_b$	Proposition 7.15
	$k = 1$, $a = b$	S_c	T_a	
	$k = 1$, $a \neq b$	S_c	S_c	
D_k		L	L	Proposition 7.16
$\mathsf{D}_k \cap \Omega_{a*}$		L_0	L_a	Proposition 7.18
$\mathsf{D}_k \cap \Omega_{*a}$		L_1	L_a	Proposition 7.18
$\mathsf{D}_k \cap \Omega_{=}$	$k \geq 2$	L_c	L	Proposition 7.19
	$k = 1$	LS	L	
$\mathsf{D}_k \cap \Omega_{\neq}$	$k \geq 2$	L_c	LS	Proposition 7.20
	$k = 1$	LS	LS	
$\mathsf{D}_k \cap \Omega_{ab}$		L_c	$\mathsf{L}_a \cap \mathsf{L}_b$	Proposition 7.21
$\mathsf{D}_i \cap \mathsf{X}_j$		LS	L	Proposition 7.22
$\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$		L_c	L_a	Proposition 7.23
$\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$		L_c	L_a	Proposition 7.23
$\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}$	$j \geq 2$	L_c	L	Proposition 7.24
	$j = 1$	LS	L	
$\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$	$j \geq 2$	L_c	LS	Proposition 7.25
	$j = 1$	LS	LS	
$\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$		L_c	$\mathsf{L}_a \cap \mathsf{L}_b$	Proposition 7.26
D_0		Ω	Ω	Proposition 7.17
$\mathsf{D}_0 \cap \Omega_{a*}$		Ω	T_a	Proposition 7.17
$\emptyset$		Ω	Ω	Proposition 7.4

TABLE 2. The L_c-stable classes K and their stability under right and left compositions with clones C. Parameters: $a, b \in \{0, 1\}$, $i, j, k \in \mathbb{N}$ with $k \geq 1$, $i > j \geq 1$.

(ii) Let

$$f_0 := x_1 + x_2 + a, \qquad g_0 := W_i + a, \qquad h_0 := x_1 \dots x_j + x_{j+1} + a,$$
$$f_1 := x_1 + a, \qquad g_1 := W_i + x_{i+1} + a, \qquad h_1 := x_1 \dots x_j + a,$$

and note that $f_0, g_0, h_0 \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{aa}$ and $f_1, g_1, h_1 \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a\overline{a}}$.

(d) Clearly for any $a, b \in \{0, 1\}$ and for any $f \in \Omega_{a*} \cup \Omega_{*b}$ we have $(x_1 + 1)(f) = f + 1 \notin \Omega_{a*} \cup \Omega_{*b}$. For any $a, b \in \{0, 1\}$ there exists a function in $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$; consider the functions f_0 and f_1 defined above. It follows that $\mathsf{I}^* K \nsubseteq \Omega_{a*} \cup \Omega_{*b}$.

(e) The reduced polynomial of each of the functions

$$\wedge(W_i + a, x_{i+1} + x_{i+2} + a), \qquad \wedge(W_i + x_{i+1} + a, x_{i+1} + a),$$
$$\vee(W_i + a, x_{i+1} + x_{i+2} + a), \qquad \vee(W_i + x_{i+1} + a, x_{i+1} + a),$$

contains the monomial $x_1 x_2 \dots x_{i+1}$ and hence has degree at least $i + 1$; therefore none of them is an element of D_i. Note that the inner functions of the two compositions on the left (right, resp.) are minors of f_0 and g_0 (f_1 and g_0, resp.) and hence belong to K if $a = b$ (if $a \neq b$, resp.). This shows that $\mathsf{\Lambda}_{\mathrm{c}} K \nsubseteq \mathsf{D}_i$, $\mathsf{V}_{\mathrm{c}} K \nsubseteq \mathsf{D}_i$.

If $j \geq 2$, then

$$\wedge(h_0, x_{j+1} + x_{j+2} + a) = x_1 \dots x_j x_{j+1} + x_1 \dots x_j x_{j+2} + \dots,$$
$$\vee(h_0, x_{j+1} + x_{j+2} + a) = x_1 \dots x_j x_{j+1} + x_1 \dots x_j x_{j+2} + \dots,$$
$$\wedge(h_1, x_{j+1} + a) = x_1 \dots x_j x_{j+1} + \dots,$$
$$\vee(h_1, x_{j+1} + a) = x_1 \dots x_j x_{j+1} + \dots,$$

where the terms that have not been written out have degree at most j. The j-element set $\{2, \dots, j + 1\}$ has characteristic 1 in each, so these functions are not in X_j. Note that the inner functions of the first (last, resp.) two compositions are minors of h_0 and f_0 (h_1 and f_1, resp.) and hence belong to K if $a = b$ (if $a \neq b$, resp.). This shows that $\mathsf{\Lambda}_{\mathrm{c}} K \nsubseteq \mathsf{X}_j$, $\mathsf{V}_{\mathrm{c}} K \nsubseteq \mathsf{X}_j$ if $j \geq 2$.

If $j = 1$ and $a \neq b$, then

$$\wedge(x_1 + a, x_2 + a) = x_1 x_2 + a x_1 + a x_2 + a \notin \mathsf{X}_1,$$
$$\vee(x_1 + a, x_2 + a) = x_1 x_2 + (a + 1) x_1 + (a + 1) x_2 + a \notin \mathsf{X}_1.$$

Note that the inner functions are minors of f_1 and hence belong to K. This shows that $\mathsf{\Lambda}_{\mathrm{c}} K \nsubseteq \mathsf{X}_j$, $\mathsf{V}_{\mathrm{c}} K \nsubseteq \mathsf{X}_j$ also in this case.

(f) The reduced polynomial of each of the functions

$$\mu(W_i + a, x_{i+1} + x_{i+2} + a, a), \qquad \mu(W_i + x_{i+1} + a, x_{i+1} + a, x_{i+2} + a)$$

contains the monomial $x_1 x_2 \dots x_{i+1}$ and hence has degree at least $i + 1$; therefore none of them is an element of D_i. Note that the inner functions of the two compositions on the left (right, resp.) are minors of f_0 and g_0 (f_1 and g_0, resp.) and hence belong to K if $a = b$ (if $a \neq b$, resp.). This shows that $\mathsf{SM} K \nsubseteq \mathsf{D}_i$.

If $j \geq 2$, then

$$\mu(h_0, x_{j+1} + x_{j+2} + a, a) = x_1 \dots x_j x_{j+1} + x_1 \dots x_j x_{j+2} + x_{j+1} + x_{j+1} x_{j+2} + a,$$
$$\mu(h_1, x_{j+1} + a, x_{j+2} + a) = x_1 \dots x_j x_{j+1} + x_1 \dots x_j x_{j+2} + x_{j+1} x_{j+2} + a.$$

Neither of these functions is in X_j, which can be seen by considering the characteristic of the j-element set $\{2, \dots, j + 1\}$. Note that the inner functions of the first (second, resp.) composition are minors of h_0 and f_0 (h_1 and f_1, resp.) and hence belong to K if $a = b$ (if $a \neq b$, resp.). This shows that $\mathsf{SM} K \nsubseteq \mathsf{X}_j$ if $j \geq 2$.

(g) If $a = b$, then

$$f_0(x_1, 0) = x_1 + a \notin \Omega_{*b}, \qquad f_0(x_1, 1) = x_1 + a + 1 \notin \Omega_{a*},$$
$$f_0(x_1 + 1, x_2) = x_1 + x_2 + a + 1 \notin \Omega_{a*} \cup \Omega_{*b}.$$

If $a \neq b$, then

$$f_1(0) = a \notin \Omega_{*b}, \qquad f_1(1) = a + 1 \notin \Omega_{a*}, \qquad f_1(x_1 + 1) = x_1 + a + 1 \notin \Omega_{a*} \cup \Omega_{*b}.$$

These calculations show the non-inclusions $K\mathsf{I}_0 \nsubseteq \Omega_{*b}$, $K\mathsf{I}_1 \nsubseteq \Omega_{a*}$, and $K\mathsf{I}^* \nsubseteq \Omega_{a*} \cup \Omega_{*b}$.

(h) Assume that $i > j$. Observe that each one of the functions $g_0(x_1, \dots, x_i, 0)$, $g_0(x_1, \dots, x_i, 1)$, $g_1(x_1, \dots, x_i, 0, 0)$, and $g_1(x_1, \dots, x_i, 1, 1)$ contains the monomial $x_1 \dots x_i$, and it is the only monomial of degree i. Therefore none of them is a member of X_j, which can be seen by considering the characteristic of the set $[i-1]$ that has cardinality at least j. We conclude that $K\mathsf{I}_0 \nsubseteq \mathsf{X}_j$ and $K\mathsf{I}_1 \nsubseteq \mathsf{X}_j$.

(i) For $i \in \{0, 1\}$, the reduced polynomial of each of the functions $g_i * \wedge$, $g_i * \vee$ contains the monomial $x_1 x_2 \dots x_{i+1}$ and hence has degree at least (in fact, exactly) $i + 1$; therefore none of them is an element of D_i. Therefore $K\mathsf{\Lambda}_{\mathrm{c}} \nsubseteq \mathsf{D}_i$, $K\mathsf{V}_{\mathrm{c}} \nsubseteq \mathsf{D}_i$.

For $i \in \{0, 1\}$, the reduced polynomial of each of $h_i * \wedge$, $h_i * \vee$ contains the monomial $x_1 \dots x_{j+1}$, and this is the only monomial of degree $j + 1$. We see that the characteristic of the j-element set $[j]$ is 1 in each, so none is an element of X_j; therefore $K\mathsf{\Lambda}_{\mathrm{c}} \nsubseteq \mathsf{X}_j$, $K\mathsf{V}_{\mathrm{c}} \nsubseteq \mathsf{X}_j$.

(j) For $i \in \{0, 1\}$, the reduced polynomial of $g_i * \mu$ contains the monomial $x_1 x_2 \dots x_{i+1}$ and hence has degree at least (in fact, exactly) $i + 1$; therefore $g_i * \mu \notin \mathsf{D}_i$. Therefore $K\mathsf{SM} \nsubseteq \mathsf{D}_i$,

If $j \geq 2$, then

$$\begin{aligned} h_0 * \mu &= x_1 x_2 x_4 \dots x_{j+2} + x_1 x_3 x_4 \dots x_{j+2} + x_2 x_3 x_4 \dots x_{j+2} + x_{j+3} + a, \\ h_1 * \mu &= x_1 x_2 x_4 \dots x_{j+2} + x_1 x_3 x_4 \dots x_{j+2} + x_2 x_3 x_4 \dots x_{j+2} + a, \end{aligned}$$

so the characteristic of the j-element set $\{1, \dots, j+1\} \setminus \{3\}$ is 1. Therefore, for $i \in \{0, 1\}$, $h_i * \mu \notin \mathsf{X}_j$, which shows that $K\mathsf{SM} \nsubseteq \mathsf{X}_j$.

(iii) (k) The following calculations show that $\mathsf{SM}(\mathsf{X}_1 \cap \Omega_{a*}) \nsubseteq \mathsf{X}_1$ (the first line) and $\mathsf{SM}(\mathsf{X}_1 \cap \Omega_{*a}) \nsubseteq \mathsf{X}_1$ (the second line) for $a \in \{0, 1\}$:

$$\begin{aligned} &\mu(x_1, x_2, 0) = x_1 x_2, && \mu(x_1 + 1, x_2 + 1, 1) = x_1 x_2 + 1, \\ &\mu(x_1, x_2, 1) = x_1 x_2 + x_1 + x_2, && \mu(x_1 + 1, x_2 + 1, 0) = x_1 x_2 + x_1 + x_2 + 1. \end{aligned}$$

(iv) (l) We have $f := x_1 + x_2 \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=$ and $f' := x_1 \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$, but $f(x_1, 0) = x_1 \notin \Omega_=$, $f'(0) = 0 \notin \Omega_{\neq}$, $f(x_1, 1) = x_1 + 1 \notin \Omega_=$, $f'(1) = 1 \notin \Omega_{\neq}$, which shows that $K\mathsf{I}_0 \nsubseteq \Omega_{\approx}$ and $K\mathsf{I}_1 \nsubseteq \Omega_{\approx}$.

(m) Assume that $j \geq 2$. We have $g := x_1 x_2 + x_2 \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=$ and $g' := x_1 x_2 \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$, but $g(x_1, x_2 + 1) = x_1 x_2 + x_1 + x_2 \notin \Omega_=$ and $g'(x_1, x_2 + 1) = x_1 x_2 + x_1 \notin \Omega_{\neq}$; therefore $K\mathsf{I}^* \nsubseteq \Omega_{\approx}$.

(n) We have $x_1, x_1 + 1 \in \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$, but $\wedge(x_1, x_1 + 1) = x_1 \cdot (x_1 + 1) = x_1 + x_1 = 0 \notin \Omega_{\neq}$, $\vee(x_1, x_1 + 1) = x_1 \cdot (x_1 + 1) + x_1 + (x_1 + 1) = 1 \notin \Omega_{\neq}$; therefore $\mathsf{\Lambda}_{\mathrm{c}} K \nsubseteq \Omega_{\neq}$ and $\mathsf{V}_{\mathrm{c}} K \nsubseteq \Omega_{\neq}$. □

Proposition 7.4. *For every clone C, we have $\Omega C \subseteq \Omega$, $C\Omega \subseteq \Omega$, $\emptyset C \subseteq \emptyset$, $C\emptyset \subseteq \emptyset$.*

Proof. Trivial. □

Proposition 7.5. *Let $a \in \{0, 1\}$, and let C be a clone.*

(i) *$\Omega_{a*} C \subseteq \Omega_{a*}$ if and only if $C \subseteq \mathsf{T}_0$.*
(ii) *$C \Omega_{a*} \subseteq \Omega_{a*}$ if and only if $C \subseteq \mathsf{T}_a$.*
(iii) *$\Omega_{*a} C \subseteq \Omega_{*a}$ if and only if $C \subseteq \mathsf{T}_1$.*
(iv) *$C \Omega_{*a} \subseteq \Omega_{*a}$ if and only if $C \subseteq \mathsf{T}_a$.*

Proof. (i) Assume first that $C \subseteq \mathsf{T}_0$. For any $f \in \Omega_{a*}^{(n)}$ and $g_1, \dots, g_n \in C^{(m)}$, we have

$$f(g_1, \dots, g_n)(0, \dots, 0) = f(g_1(0, \dots, 0), \dots, g_n(0, \dots, 0)) = f(0, \dots, 0) = a,$$

so $f(g_1, \dots, g_n) \in \Omega_{a*}$. We conclude that $\Omega_{a*} C \subseteq \Omega_{a*}$. Conversely, if $\Omega_{a*} C \subseteq \Omega_{a*}$, then C includes neither I_1 nor I^* by Lemma 7.3(g), so $C \subseteq \mathsf{T}_0$.

(ii) Assume first that $C \subseteq \mathsf{T}_a$. For any $f \in C^{(n)}$ and $g_1, \dots, g_n \in \Omega_{a*}^{(m)}$, we have

$$f(g_1, \dots, g_n)(0, \dots, 0) = f(g_1(0, \dots, 0), \dots, g_n(0, \dots, 0)) = f(a, \dots a) = a,$$

so $f(g_1, \dots g_n) \in \Omega_{a*}$. We conclude that $C\Omega_{a*} \subseteq \Omega_{a*}$. Conversely, if $C\Omega_{a*} \subseteq \Omega_{a*}$, then C includes neither $\mathsf{I}_{\overline{a}}$ nor I^* by Lemma 7.3(a), (d), so $C \subseteq \mathsf{T}_a$.

(iii) Assume first that $C \subseteq \mathsf{T}_1$. For any $f \in \Omega_{*a}^{(n)}$ and $g_1, \dots, g_n \in C^{(m)}$, we have

$$f(g_1, \dots, g_n)(1, \dots, 1) = f(g_1(1, \dots, 1), \dots, g_n(1, \dots, 1)) = f(1, \dots, 1) = a,$$

so $f(g_1, \dots, g_n) \in \Omega_{*a}$. We conclude that $\Omega_{*a} C \subseteq \Omega_{*a}$. Conversely, if $\Omega_{*a} C \subseteq \Omega_{*a}$, then C includes neither I_0 nor I^* by Lemma 7.3(g), so $C \subseteq \mathsf{T}_1$.

(iv) Assume first that $C \subseteq \mathsf{T}_a$. For any $f \in C^{(n)}$ and $g_1, \dots, g_n \in \Omega_{*a}^{(m)}$, we have

$$f(g_1, \dots, g_n)(1, \dots, 1) = f(g_1(1, \dots, 1), \dots, g_n(1, \dots, 1)) = f(a, \dots a) = a,$$

so $f(g_1, \dots g_n) \in \Omega_{*a}$. We conclude that $C\Omega_{*a} \subseteq \Omega_{*a}$. Conversely, if $C\Omega_{*a} \subseteq \Omega_{*a}$, then C includes neither $\mathsf{I}_{\overline{a}}$ nor I^* by Lemma 7.3(a), (d), so $C \subseteq \mathsf{T}_a$. □

Lemma 7.6.

(i) *For any $f, g \in \Omega_=$, we have $f \cdot g \in \Omega_=$.*

(ii) *For any $f, g \in \Omega_{\neq}$, we have $f \cdot g \in \Omega_{\neq}$ if and only if both f and g have equal constant terms (i.e, $f, g \in \Omega_{0*}$ or $f, g \in \Omega_{1*}$).*

Proof. (i) Let $\alpha, \beta \in \Omega_= \cap \Omega_{0*}$. Then both α and β are sums of an even number of monomials. We have $\alpha \cdot \beta \in \Omega_{\neq}$ because the expansion of the product of the two even sums of monomials yields a sum of an even number of monomials. We clearly also have that $(\alpha + 1) \cdot \beta = \alpha \cdot \beta + \beta$, $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$, and $(\alpha + 1) \cdot (\beta + 1) = \alpha \cdot \beta + \alpha + \beta + 1$ belong to $\Omega_=$ because they are sums of polynomials with an even number of monomials plus a possible constant term. The claim now follows because any $f \in \Omega_=$ is of the form α or $\alpha + 1$ for some $\alpha \in \Omega_= \cap \Omega_{0*}$.

(ii) Let $\alpha, \beta \in \Omega_{\neq} \cap \Omega_{0*}$. Then both α and β are sums of an odd number of monomials. We have $\alpha \cdot \beta \in \Omega_{\neq}$ because the expansion of the product of the two odd sums of monomials yields a sum of an odd number of monomials. Consequently, $(\alpha + 1) \cdot \beta = \alpha \cdot \beta + \beta \in \Omega_=$, $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha \in \Omega_=$, and $(\alpha + 1) \cdot (\beta + 1) = \alpha \cdot \beta + \alpha + \beta + 1 \in \Omega_{\neq}$. □

Proposition 7.7. *Let C be a clone.*

(i) *$\Omega_= C \subseteq \Omega_=$ if and only if $C \subseteq \mathsf{T}_{\mathrm{c}}$.*

(ii) *$C\Omega_= \subseteq \Omega_=$ for any clone C.*

(iii) *$\Omega_{\neq} C \subseteq \Omega_{\neq}$ if and only if $C \subseteq \mathsf{T}_{\mathrm{c}}$.*

(iv) *$C\Omega_{\neq} \subseteq \Omega_{\neq}$ if and only if $C \subseteq \mathsf{S}$.*

Proof. Recall that $\mathsf{T}_{\mathrm{c}} = \Omega_{0*} \cap \Omega_{\neq}$.

(i) Assume first that $C \subseteq \mathsf{T}_{\mathrm{c}}$. Let $f \in \Omega_=^{(n)}$ and $g_1, \dots, g_n \in C^{(m)}$. Observing that $\mathsf{T}_{\mathrm{c}} = \Omega_{01} = \Omega_{\neq} \cap \Omega_{0*}$, we have

$$f(g_1, \dots, g_n) = \sum_{S \in M_f} \prod_{i \in S} g_i \in \Omega_=,$$

because each summand $\prod_{i \in S} g_i$ is odd by Lemma 7.6, and there are an even number of such summands since f is even. We conclude that $\Omega_= C \subseteq \Omega_=$. Conversely, if $\Omega_= C \subseteq \Omega_=$, then C includes neither I_0, I_1, nor I^* by Lemma 7.3(l), (m), so $C \subseteq \mathsf{T}_{\mathrm{c}}$.

(ii) It is enough to prove the claim for $C = \Omega$. Using the fact that $\Omega = \langle x_1 x_2 + 1 \rangle$, we will apply Lemma 3.3. Let $g_1, g_2 \in \Omega_=$. With the help of Lemma 7.6, we see that $(x_1 x_2 + 1)(g_1, g_2) = g_1 g_2 + 1 \in \Omega_=$. Now it follows from Lemma 3.3 that $\Omega\Omega_= \subseteq \Omega_=$.

(iii) Assume first that $C \subseteq \mathsf{T}_{\mathrm{c}}$. Let $f \in \Omega_{\neq}^{(n)}$ and $g_1, \dots, g_n \in C^{(m)}$. If $f \in \Omega_{0*}$, then $f \in \Omega_{0*} \cap \Omega_{\neq} = \mathsf{T}_{\mathrm{c}}$, and it follows immediately from the fact that T_{c} is a clone that $f(g_1, \dots, g_n) \in \mathsf{T}_{\mathrm{c}} = \Omega_{0*} \cap \Omega_{\neq} \subseteq \Omega_{\neq}$. If $f \in \Omega_{1*}$, then $f' := f + 1 \in \Omega_{0*} \cap \Omega_{\neq} = \mathsf{T}_{\mathrm{c}}$. It follows from Lemma 2.8 that

$$\begin{aligned} f(g_1, \dots, g_n) &= (f' + 1)(g_1, \dots, g_n) = f'(g_1, \dots, g_n) + 1(g_1, \dots, g_n) \\ &= \underbrace{f'(g_1, \dots, g_n)}_{\in \mathsf{T}_{\mathrm{c}} = \Omega_{0*} \cap \Omega_{\neq}} + 1 \in \Omega_{1*} \cap \Omega_{\neq} \subseteq \Omega_{\neq}. \end{aligned}$$

We conclude that $\Omega_{\neq} C \subseteq \Omega_{\neq}$. Conversely, if $\Omega_{\neq} C \subseteq \Omega_{\neq}$, then C includes neither I_0, I_1, nor I^* by Lemma 7.3(l), (m), so $C \subseteq \mathsf{T}_{\mathrm{c}}$.

(iv) For sufficiency, it is enough to prove the claim for $C = \mathsf{S}$. Using the fact that $\mathsf{S} = \langle \mu, x_1 + 1 \rangle$, we will apply Lemma 3.3. Let $g_1, g_2, g_3 \in \Omega_{\neq}$. We clearly have $(x_1 + 1)(g_1) = g_1 + 1 \in \Omega_{\neq}$. Applying Lemma 7.6, we see that $\mu(g_1, g_2, g_3) = g_1 g_2 + g_1 g_3 + g_2 g_3 \in \Omega_{\neq}$; for if g_1, g_2, g_3 have the same constant term, then the three summands $g_1 g_2$, $g_1 g_3$, $g_2 g_3$ belong to $\Omega_{\neq}$; if they do not all have the same constant term, then it is easy to see that exactly one of the summands belongs to $\Omega_{\neq}$ and the other two belong to $\Omega_{=}$. Now it follows from Lemma 3.3 that $\mathsf{S}\Omega_{\neq} \subseteq \Omega_{\neq}$.

For necessity, assume that $C\Omega_{\neq} \subseteq \Omega_{\neq}$. Then C includes neither I_0, I_1, $\mathsf{\Lambda}_{\mathrm{c}}$, nor V_{c} by Lemma 7.3(b), (n), so $C \subseteq \mathsf{S}$. □

Proposition 7.8. *Let $a, b \in \{0, 1\}$, and let C be a clone.*

(i) $(\Omega_{ab})C \subseteq (\Omega_{ab})$ *if and only if* $C \subseteq \mathsf{T}_{\mathrm{c}}$.
(ii) $C(\Omega_{ab}) \subseteq (\Omega_{ab})$ *if and only if* $C \subseteq \mathsf{T}_a \cap \mathsf{T}_b$.

Proof. (i) Lemma 7.2 and Proposition 7.5(i), (iii) imply that $(\Omega_{ab})C \subseteq \Omega_{ab}$ whenever $C \subseteq \mathsf{T}_0 \cap \mathsf{T}_1 = \mathsf{T}_{\mathrm{c}}$. Conversely, if $(\Omega_{ab})C \subseteq \Omega_{ab}$, then C includes neither I_0, I_1, nor I^* by Lemma 7.3(g), so $C \subseteq \mathsf{T}_{\mathrm{c}}$.

(ii) Lemma 7.2 and Proposition 7.5(ii), (iv) imply that $C(\Omega_{ab}) \subseteq \Omega_{ab}$ whenever $C \subseteq \mathsf{T}_a \cap \mathsf{T}_b$.

Assume now that $C(\Omega_{ab}) \subseteq \Omega_{ab}$. If $a = b$, then C includes neither $\mathsf{I}_{\overline{a}}$ nor I^* by Lemma 7.3(a), (d), so $C \subseteq \mathsf{T}_a = \mathsf{T}_a \cap \mathsf{T}_b$. If $a \neq b$, then C includes neither I_0, I_1, nor I^* by Lemma 7.3(c), (d), so $C \subseteq \mathsf{T}_{\mathrm{c}} = \mathsf{T}_a \cap \mathsf{T}_b$. □

Lemma 7.9.

(i) $\mathsf{X}_0\mathsf{S} \subseteq \mathsf{X}_0$.
(ii) $\Omega\mathsf{X}_0 \subseteq \mathsf{X}_0$.

Proof. (i) Let $f \in \mathsf{X}_0^{(n)}$ and $g_1, \dots, g_n \in \mathsf{S}^{(m)}$. Since X_0 is the class of all reflexive functions and S is the class of all self-dual functions, we have, for any $\mathbf{a} \in \{0, 1\}^m$ that

$$\begin{aligned} f(g_1, \dots, g_n)(\overline{\mathbf{a}}) &= f(g_1(\overline{\mathbf{a}}), \dots, g_n(\overline{\mathbf{a}})) = f(\overline{g_1(\mathbf{a})}, \dots, \overline{g_n(\mathbf{a})}) \\ &= f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})) = f(g_1, \dots, g_n)(\mathbf{a}), \end{aligned}$$

so $f(g_1, \dots, g_n) \in \mathsf{X}_0$.

(ii) Let $f \in \Omega^{(n)}$, $g_1, \dots, g_n \in \mathsf{X}_0^{(m)}$. We have, for any $\mathbf{a} \in \{0, 1\}^m$,

$$f(g_1, \dots, g_n)(\overline{\mathbf{a}}) = f(g_1(\overline{\mathbf{a}}), \dots, g_n(\overline{\mathbf{a}})) = f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})) = f(g_1, \dots, g_n)(\mathbf{a}),$$

so $f(g_1, \dots, g_n) \in \mathsf{X}_0$. □

Proposition 7.10. *Let $k \in \mathbb{N}_+$, and let C be a clone.*

(i) *For $k \geq 2$,* $\mathsf{X}_k C \subseteq \mathsf{X}_k$ *if and only if* $C \subseteq \mathsf{LS}$.
(ii) $\mathsf{X}_1 C \subseteq \mathsf{X}_1$ *if and only if* $C \subseteq \mathsf{S}$.
(iii) $C\mathsf{X}_k \subseteq \mathsf{X}_k$ *if and only if* $C \subseteq \mathsf{L}$.

Proof. (i) For sufficiency, it is enough to prove the claim for $C = \mathsf{LS}$. Using the fact that $\mathsf{LS} = \langle \oplus_3, x_1 + 1 \rangle$, we apply Lemma 3.2. Let $f \in \mathsf{X}_k$. We have $f * \oplus_3 = \oplus_3(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3})$, where the σ_i are as in Lemma 6.2. Since X_k is closed under minors and sums by Lemmata 6.7 and 6.8, we have $\oplus_3(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}) \in \mathsf{X}_k$. As for $f * (x_1 + 1)$, note that $f * (x_1 + 1) = f + f'_1$ by Lemma 6.17. Since $f'_1 \in \mathsf{X}_{k-1} \subseteq \mathsf{X}_k$ by Lemma 6.18, we have $f * (x_1 + 1) = f + f'_1 \in \mathsf{X}_k$ by Lemma 6.8. It follows from Lemma 3.2 that $\mathsf{X}_k \, \mathsf{LS} \subseteq \mathsf{X}_k$.

For necessity, assume that $\mathsf{X}_k C \subseteq \mathsf{X}_k$. Then C includes neither I_0, I_1, Λ_{c}, V_{c}, nor SM by Lemma 7.3(h), (i), (j), so $C \nsubseteq \mathsf{LS}$.

(ii) Assume first that $C \subseteq \mathsf{S}$. Since $\mathsf{X}_1 = (\mathsf{X}_1 \cap \Omega_{=}) \cup (\mathsf{X}_1 \cap \Omega_{\neq}) = \mathsf{X}_0 \cup \mathsf{S}$ and S is a clone, it follows from Lemmata 2.11 and 7.9(i) that

$$\mathsf{X}_1\mathsf{S} \subseteq (\mathsf{X}_0 \cup \mathsf{S})\mathsf{S} = \mathsf{X}_0\mathsf{S} \cup \mathsf{SS} \subseteq \mathsf{X}_0 \cup \mathsf{S} = \mathsf{X}_1.$$

Conversely, if $\mathsf{X}_1 C \subseteq \mathsf{X}_1$, then C includes neither I_0, I_1, Λ_{c}, nor V_{c} by Lemma 7.3(h), (i), so $C \subseteq \mathsf{S}$.

(iii) For sufficiency, it is enough to prove the claim for $C = \mathsf{L}$. Using the fact that $\mathsf{L} = \langle x_1 + x_2, 1 \rangle$, we apply Lemma 3.3. For any $g_1, g_2 \in \mathsf{X}_k^{(n)}$, we clearly have $1(g_1) = 1 \in \mathsf{X}_k$ and $(x_1 + x_2)(g_1, g_2) = g_1 + g_2 \in \mathsf{X}_k$ by Lemma 6.8. It follows from Lemma 3.3 that $\mathsf{L}\mathsf{X}_k \subseteq \mathsf{X}_k$.

For necessity, assume that $C\mathsf{X}_k \subseteq \mathsf{X}_k$. Then C includes neither Λ_{c}, V_{c}, nor SM by Lemma 7.3(e), (f), (k) so $C \subseteq \mathsf{L}$. $\square$

Proposition 7.11. *Let $k \in \mathbb{N}_+$, $a \in \{0, 1\}$, and let C be a clone.*

(i) *For $k \geq 2$, $(\mathsf{X}_k \cap \Omega_{a*})C \subseteq \mathsf{X}_k \cap \Omega_{a*}$ if and only if $C \subseteq \mathsf{L}_{\mathrm{c}}$.*
(ii) *$(\mathsf{X}_1 \cap \Omega_{a*})C \subseteq \mathsf{X}_1 \cap \Omega_{a*}$ if and only if $C \subseteq \mathsf{S}_{\mathrm{c}}$.*
(iii) *$C(\mathsf{X}_k \cap \Omega_{a*}) \subseteq \mathsf{X}_k \cap \Omega_{a*}$ if and only if $C \subseteq \mathsf{L}_a$.*

Proof. (i) Lemma 7.2 and Propositions 7.5(i) and 7.10(i) imply that $(\mathsf{X}_k \cap \Omega_{a*})C \subseteq \mathsf{X}_k \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{T}_0 \cap \mathsf{LS} = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_k \cap \Omega_{a*})C \subseteq \mathsf{X}_k \cap \Omega_{a*}$, then C includes neither I_0, I_1, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(g), (h), (i), (j), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) Lemma 7.2 and Propositions 7.5(i) and 7.10(i) imply that $(\mathsf{X}_1 \cap \Omega_{a*})C \subseteq \mathsf{X}_1 \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{T}_0 \cap \mathsf{S} = \mathsf{S}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_1 \cap \Omega_{a*})C \subseteq \mathsf{X}_1 \cap \Omega_{a*}$, then C includes neither I_0, I_1, I^*, Λ_{c}, nor V_{c} by Lemma 7.3(g), (h), (i), so $C \subseteq \mathsf{S}_{\mathrm{c}}$.

(iii) Lemma 7.2 and Propositions 7.5(ii) and 7.10(iii) imply that $C(\mathsf{X}_k \cap \Omega_{a*}) \subseteq \mathsf{X}_k \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{T}_a \cap \mathsf{L} = \mathsf{L}_a$. Conversely, if $C(\mathsf{X}_k \cap \Omega_{a*}) \subseteq \mathsf{X}_k \cap \Omega_{a*}$, then C includes neither $\mathsf{I}_{\overline{a}}$, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(a), (d), (e), (f), (k), so $C \subseteq \mathsf{L}_a$. $\square$

Proposition 7.12. *Let $k \in \mathbb{N}_+$, $a \in \{0, 1\}$, and let C be a clone.*

(i) *For $k \geq 2$, $(\mathsf{X}_k \cap \Omega_{*a})C \subseteq \mathsf{X}_k \cap \Omega_{*a}$ if and only if $C \subseteq \mathsf{L}_{\mathrm{c}}$.*
(ii) *$(\mathsf{X}_1 \cap \Omega_{*a})C \subseteq \mathsf{X}_1 \cap \Omega_{*a}$ if and only if $C \subseteq \mathsf{S}_{\mathrm{c}}$.*
(iii) *$C(\mathsf{X}_k \cap \Omega_{*a}) \subseteq \mathsf{X}_k \cap \Omega_{*a}$ if and only if $C \subseteq \mathsf{L}_a$.*

Proof. (i) Lemma 7.2 and Propositions 7.5(iii) and 7.10(i) imply that $(\mathsf{X}_k \cap \Omega_{*a})C \subseteq \mathsf{X}_k \cap \Omega_{*a}$ whenever $C \subseteq \mathsf{T}_1 \cap \mathsf{LS} = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_k \cap \Omega_{*a})C \subseteq \mathsf{X}_k \cap \Omega_{*a}$, then C includes neither I_0, I_1, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(g), (h), (i), (j), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) Lemma 7.2 and Propositions 7.5(iii) and 7.10(i) imply that $(\mathsf{X}_1 \cap \Omega_{*a})C \subseteq \mathsf{X}_1 \cap \Omega_{*a}$ whenever $C \subseteq \mathsf{T}_1 \cap \mathsf{S} = \mathsf{S}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_1 \cap \Omega_{*a})C \subseteq \mathsf{X}_1 \cap \Omega_{*a}$, then C includes neither I_0, I_1, I^*, Λ_{c}, nor V_{c} by Lemma 7.3(g), (h), (i), so $C \subseteq \mathsf{S}_{\mathrm{c}}$.

(iii) Lemma 7.2 and Propositions 7.5(iv) and 7.10(iii) imply that $C(\mathsf{X}_k \cap \Omega_{a*}) \subseteq \mathsf{X}_k \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{T}_a \cap \mathsf{L} = \mathsf{L}_a$. Conversely, if $C(\mathsf{X}_k \cap \Omega_{*a}) \subseteq \mathsf{X}_k \cap \Omega_{*a}$, then

C includes neither $\mathsf{I}_{\overline{a}}$, I^*, $\mathsf{\Lambda}_{\mathrm{c}}$, V_{c}, nor SM by Lemma 7.3(a), (d), (e), (f), (k), so $C \subseteq \mathsf{L}_a$. □

Proposition 7.13. *Let $k \in \mathbb{N}_+$, and let C be a clone.*

(i) *For $k \geq 2$, $(\mathsf{X}_k \cap \Omega_=)C \subseteq \mathsf{X}_k \cap \Omega_=$ if and only if $C \subseteq \mathsf{L}_{\mathrm{c}}$.*
(ii) *$(\mathsf{X}_1 \cap \Omega_=)C \subseteq \mathsf{X}_1 \cap \Omega_=$ if and only if $C \subseteq \mathsf{S}$.*
(iii) *For $k \geq 2$, $C(\mathsf{X}_k \cap \Omega_=) \subseteq \mathsf{X}_k \cap \Omega_=$ if and only if $C \subseteq \mathsf{L}$.*
(iv) *$C(\mathsf{X}_1 \cap \Omega_=) \subseteq \mathsf{X}_1 \cap \Omega_=$ for any clone C.*

Proof. (i) Lemma 7.2 and Propositions 7.7(i) and 7.10(i) imply that $(\mathsf{X}_k \cap \Omega_=)C \subseteq \mathsf{X}_k \cap \Omega_=$ whenever $C \subseteq \mathsf{L}_{\mathrm{c}} \cap \mathsf{T}_{\mathrm{c}} = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_k \cap \Omega_=)C \subseteq \mathsf{X}_k \cap \Omega_=$, then C includes neither I_0, I_1, I^*, $\mathsf{\Lambda}_{\mathrm{c}}$, V_{c}, nor SM by Lemma 7.3(i), (j), (l), (m), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) Assume first that $C \subseteq \mathsf{S}$. Since $\mathsf{X}_1 \cap \Omega_= = \mathsf{X}_0$, it follows from Lemma 7.9(i) that $(\mathsf{X}_1 \cap \Omega_=)C \subseteq \mathsf{X}_0\mathsf{S} \subseteq \mathsf{X}_0 = \mathsf{X}_1 \cap \Omega_=$. Conversely, if $(\mathsf{X}_1 \cap \Omega_=)C \subseteq \mathsf{X}_1 \cap \Omega_=$, then C includes neither I_0, I_1, $\mathsf{\Lambda}_{\mathrm{c}}$, nor V_{c} by Lemma 7.3(i), (l), so $C \subseteq \mathsf{S}$.

(iii) Lemma 7.2 and Propositions 7.7(ii) and 7.10(iii) imply that $C(\mathsf{X}_k \cap \Omega_=) \subseteq \mathsf{X}_k \cap \Omega_=$ whenever $C \subseteq \mathsf{L} \cap \Omega = \mathsf{L}$. Conversely, if $C(\mathsf{X}_k \cap \Omega_=) \subseteq \mathsf{X}_k \cap \Omega_=$, then C includes neither $\mathsf{\Lambda}_{\mathrm{c}}$, V_{c}, nor SM by Lemma 7.3(e), (f) (note that $\Omega_{00} \subseteq \Omega_=$), so $C \subseteq \mathsf{L}$.

(iv) Observing that $\mathsf{X}_1 \cap \Omega_= = \mathsf{X}_0$, this follows immediately from Lemma 7.9(ii). □

Proposition 7.14. *Let $k \in \mathbb{N}_+$, and let C be a clone.*

(i) *For $k \geq 2$, $(\mathsf{X}_k \cap \Omega_{\neq})C \subseteq \mathsf{X}_k \cap \Omega_{\neq}$ if and only if $C \subseteq \mathsf{L}_{\mathrm{c}}$.*
(ii) *$(\mathsf{X}_1 \cap \Omega_{\neq})C \subseteq \mathsf{X}_1 \cap \Omega_{\neq}$ if and only if $C \subseteq \mathsf{S}$.*
(iii) *For $k \geq 2$, $C(\mathsf{X}_k \cap \Omega_{\neq}) \subseteq \mathsf{X}_k \cap \Omega_{\neq}$ if and only if $C \subseteq \mathsf{LS}$.*
(iv) *$C(\mathsf{X}_1 \cap \Omega_{\neq}) \subseteq \mathsf{X}_1 \cap \Omega_{\neq}$ if and only if $C \subseteq \mathsf{S}$.*

Proof. (i) Lemma 7.2 and Propositions 7.7(iii) and 7.10(i) imply that $(\mathsf{X}_k \cap \Omega_{\neq})C \subseteq \mathsf{X}_k \cap \Omega_{\neq}$ whenever $C \subseteq \mathsf{LS} \cap \mathsf{T}_{\mathrm{c}} = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_k \cap \Omega_{\neq})C \subseteq \mathsf{X}_k \cap \Omega_{\neq}$, then C includes neither I_0, I_1, I^*, $\mathsf{\Lambda}_{\mathrm{c}}$, V_{c}, nor SM by Lemma 7.3(i), (j), (l), (m), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) Assume first that $C \subseteq \mathsf{S}$. Since $\mathsf{X}_1 \cap \Omega_{\neq} = \mathsf{S}$ and S is a clone, it is immediately obvious that $(\mathsf{X}_1 \cap \Omega_{\neq})C \subseteq \mathsf{SS} \subseteq \mathsf{S} = \mathsf{X}_1 \cap \Omega_{\neq}$. Conversely, if $(\mathsf{X}_1 \cap \Omega_{\neq})C \subseteq \mathsf{X}_1 \cap \Omega_{\neq}$, then C includes neither I_0, I_1, $\mathsf{\Lambda}_{\mathrm{c}}$, nor V_{c} by Lemma 7.3(i), (l), so $C \subseteq \mathsf{S}$.

(iii) Lemma 7.2 and Propositions 7.7(iv) and 7.10(iii) imply that $C(\mathsf{X}_k \cap \Omega_{\neq}) \subseteq \mathsf{X}_k \cap \Omega_{\neq}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{S} = \mathsf{LS}$. Conversely, if $C(\mathsf{X}_k \cap \Omega_{\neq}) \subseteq \mathsf{X}_k \cap \Omega_{\neq}$, then C includes neither I_0, I_1, $\mathsf{\Lambda}_{\mathrm{c}}$, V_{c}, nor SM by Lemma 7.3(b), (e), (f), so $C \subseteq \mathsf{LS}$.

(iv) Assume first that $C \subseteq \mathsf{S}$. Since $\mathsf{X}_1 \cap \Omega_{\neq} = \mathsf{S}$ and S is a clone, it is immediately obvious that $C(\mathsf{X}_1 \cap \Omega_{\neq}) \subseteq \mathsf{SS} \subseteq \mathsf{S} = \mathsf{X}_1 \cap \Omega_{\neq}$. Conversely, if $C(\mathsf{X}_1 \cap \Omega_{\neq}) \subseteq \mathsf{X}_1 \cap \Omega_{\neq}$, then C includes neither I_0, I_1, $\mathsf{\Lambda}_{\mathrm{c}}$, nor V_{c} by Lemma 7.3(b), (e), so $C \subseteq \mathsf{S}$. □

Proposition 7.15. *Let $k \in \mathbb{N}_+$, $a, b \in \{0, 1\}$, and let C be a clone.*

(i) *For $k \geq 2$, $(\mathsf{X}_k \cap \Omega_{ab})C \subseteq \mathsf{X}_k \cap \Omega_{ab}$ if and only if $C \subseteq \mathsf{L}_{\mathrm{c}}$.*
(ii) *$(\mathsf{X}_1 \cap \Omega_{ab})C \subseteq \mathsf{X}_1 \cap \Omega_{ab}$ if and only if $C \subseteq \mathsf{S}_{\mathrm{c}}$.*
(iii) *If $k \geq 2$, then $C(\mathsf{X}_k \cap \Omega_{ab}) \subseteq \mathsf{X}_k \cap \Omega_{ab}$ if and only if $C \subseteq \mathsf{L}_a \cap \mathsf{L}_b$.*
(iv) *If $a = b$, then $C(\mathsf{X}_1 \cap \Omega_{ab}) \subseteq \mathsf{X}_1 \cap \Omega_{ab}$ if and only if $C \subseteq \mathsf{T}_a$.*
(v) *If $a \neq b$, then $C(\mathsf{X}_1 \cap \Omega_{ab}) \subseteq \mathsf{X}_1 \cap \Omega_{ab}$ if and only if $C \subseteq \mathsf{S}_{\mathrm{c}}$.*

Proof. (i) Lemma 7.2 and Propositions 7.8(i) and 7.10(i) imply that $(\mathsf{X}_k \cap \Omega_{ab})C \subseteq \mathsf{X}_k \cap \Omega_{ab}$ whenever $C \subseteq \mathsf{LS} \cap \mathsf{T}_{\mathrm{c}} = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_k \cap \Omega_{ab})C \subseteq \mathsf{X}_k \cap \Omega_{ab}$, then C includes neither I_0, I_1, I^*, $\mathsf{\Lambda}_{\mathrm{c}}$, V_{c}, nor SM by Lemma 7.3(g), (i), (j), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) Lemma 7.2 and Propositions 7.8(i) and 7.10(ii) imply that $(\mathsf{X}_1 \cap \Omega_{ab})C \subseteq \mathsf{X}_1 \cap \Omega_{ab}$ whenever $C \subseteq \mathsf{S} \cap \mathsf{T}_{\mathrm{c}} = \mathsf{S}_{\mathrm{c}}$. Conversely, if $(\mathsf{X}_1 \cap \Omega_{ab})C \subseteq \mathsf{X}_1 \cap \Omega_{ab}$, then C includes neither I_0, I_1, I^*, $\mathsf{\Lambda}_{\mathrm{c}}$, nor V_{c} by Lemma 7.3(g), (i), so $C \subseteq \mathsf{S}_{\mathrm{c}}$.

(iii) Lemma 7.2 and Propositions 7.8(ii) and 7.10(iii) imply that $C(\mathsf{X}_k \cap \Omega_{ab}) \subseteq \mathsf{X}_k \cap \Omega_{ab}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_a \cap \mathsf{T}_b = \mathsf{L}_a \cap \mathsf{L}_b$.

Assume now that $C(\mathsf{X}_k \cap \Omega_{ab}) \subseteq \mathsf{X}_k \cap \Omega_{ab}$. Then C includes neither I^*, Λ_c, V_c, nor SM by Lemma 7.3(d), (e), (f). If $a \neq b$, then C includes neither I_0 nor I_1 by Lemma 7.3(c), so $C \subseteq \mathsf{L}_\mathrm{c} = \mathsf{L}_a \cap \mathsf{L}_b$. If $a = b$, then C does not include $\mathsf{I}_{\overline{a}}$ by Lemma 7.3(a), so $C \subseteq \mathsf{L}_a = \mathsf{L}_a \cap \mathsf{L}_b$.

(iv) Assume first that $C \subseteq \mathsf{T}_a$. We have $C(\mathsf{X}_1 \cap \Omega_{aa}) \subseteq \mathsf{T}_a(\mathsf{X}_0 \cap \Omega_{a*}) \subseteq \mathsf{X}_0 \cap \Omega_{a*} = \mathsf{X}_1 \cap \Omega_{aa}$, where the second inclusion holds because $\mathsf{T}_a(\mathsf{X}_0 \cap \Omega_{a*}) \subseteq \Omega \mathsf{X}_0 \subseteq \mathsf{X}_0$ by Lemma 7.9(ii) and $\mathsf{T}_a(\mathsf{X}_0 \cap \Omega_{a*}) \subseteq \mathsf{T}_a \Omega_{a*} \subseteq \Omega_{a*}$ as can be easily seen.

Assume now that $C(\mathsf{X}_1 \cap \Omega_{aa}) \subseteq \mathsf{X}_1 \cap \Omega_{aa}$. Then C includes neither $\mathsf{I}_{\overline{a}}$ nor I^* by Lemma 7.3(a), (d), so $C \subseteq \mathsf{T}_a$.

(v) Assume first that $C \subseteq \mathsf{S}_\mathrm{c}$. Since $\mathsf{X}_1 \cap \Omega_{01} = \mathsf{X}_1 \cap \Omega_{\neq} \cap \Omega_{0*} = \mathsf{S} \cap \Omega_{0*} = \mathsf{S}_\mathrm{c}$, we have

$$C(\mathsf{X}_1 \cap \Omega_{01}) \subseteq \mathsf{S}_\mathrm{c} \mathsf{S}_\mathrm{c} \subseteq \mathsf{S}_\mathrm{c} = \mathsf{X}_1 \cap \Omega_{01}.$$

Note also that $\mathsf{X}_1 \cap \Omega_{10} = \mathsf{X}_1 \cap \Omega_{\neq} \cap \Omega_{1*} = \mathsf{S} \cap \Omega_{1*} = \mathsf{S} \setminus \mathsf{S}_\mathrm{c}$. For any $f \in \mathsf{S}_\mathrm{c}^{(n)}$, $g_1, \dots, g_n \in (\mathsf{S} \setminus \mathsf{S}_\mathrm{c})^{(m)}$, it holds that $f(g_1, \dots, g_1) \in \mathsf{S}$ and

$$f(g_1, \dots, g_n)(0, \dots, 0) = f(g_1(0, \dots, 0), \dots, g_n(0, \dots, 0)) = f(1, \dots, 1) = 1,$$

so $f(g_1, \dots, g_n) \notin \mathsf{S}_\mathrm{c}$, that is, $f(g_1, \dots, g_n) \in \mathsf{S} \setminus \mathsf{S}_\mathrm{c}$. Consequently, $\mathsf{S}_\mathrm{c}(\mathsf{S} \setminus \mathsf{S}_\mathrm{c}) \subseteq \mathsf{S} \setminus \mathsf{S}_\mathrm{c}$, and it follows that

$$C(\mathsf{X}_1 \cap \Omega_{10}) \subseteq \mathsf{S}_\mathrm{c}(\mathsf{S} \setminus \mathsf{S}_\mathrm{c}) \subseteq \mathsf{S} \setminus \mathsf{S}_\mathrm{c} = \mathsf{X}_1 \cap \Omega_{10}.$$

Assume now that $C(\mathsf{X}_1 \cap \Omega_{ab}) \subseteq \mathsf{X}_1 \cap \Omega_{ab}$. Then C includes neither I_0, I_1, I^*, Λ_c, nor V_c by Lemma 7.3(c), (d), (e), so $C \subseteq \mathsf{S}_\mathrm{c}$. $\square$

Proposition 7.16. *Let $k \in \mathbb{N}_+$, and let C be a clone.*

(i) $\mathsf{D}_k C \subseteq \mathsf{D}_k$ *if and only if* $C \subseteq \mathsf{L}$.
(ii) $C\mathsf{D}_k \subseteq \mathsf{D}_k$ *if and only if* $C \subseteq \mathsf{L}$.

Proof. (i) For sufficiency, it is enough to prove the claim for $C = \mathsf{L}$. Using the fact that $\mathsf{L} = \langle x_1 + x_2, 1 \rangle$, we apply Lemma 3.2. It is easy to see that for any function $f \in \mathsf{D}_k$, we have $\deg(f * (x_1 + x_2)) \leq \deg(f) \leq k$ and $\deg(f * 1) \leq \deg(f) \leq k$, so $f * (x_1 + x_2), f * 1 \in \mathsf{D}_k$. It follows from Lemma 3.2 that $\mathsf{D}_k \mathsf{L} \subseteq \mathsf{D}_k$.

For necessity, assume that $\mathsf{D}_k C \subseteq \mathsf{D}_k$. Then C includes neither Λ_c, V_c, nor SM by Lemma 7.3(i), (j), so $C \subseteq \mathsf{L}$.

(ii) For sufficiency, it is enough to prove the claim for $C = \mathsf{L}$. Using the fact that $\mathsf{L} = \langle x_1 + x_2, 1 \rangle$, we apply Lemma 3.3. It is clear that for any $g_1, g_2 \in \mathsf{D}_k^{(m)}$, the functions $(x_1 + x_2)(g_1, g_2) = g_1 + g_2$ and $1(g_1) = 1$ have degree at most k, and are therefore members of D_k. It follows from Lemma 3.3 that $\mathsf{L}\mathsf{D}_k \subseteq \mathsf{D}_k$.

For necessity, assume that $C\mathsf{D}_k \subseteq \mathsf{D}_k$. Then C includes neither Λ_c, V_c, nor SM by Lemma 7.3(e), (f), so $C \subseteq \mathsf{L}$. $\square$

Proposition 7.17. *Let $a \in \{0, 1\}$, and let C be a clone.*

(i) $\mathsf{D}_0 C \subseteq \mathsf{D}_0$ *and* $C\mathsf{D}_0 \subseteq \mathsf{D}_0$ *for any clone C.*
(ii) $(\mathsf{D}_0 \cap \Omega_{a*})C \subseteq \mathsf{D}_0 \cap \Omega_{a*}$ *for any clone C.*
(iii) $C(\mathsf{D}_0 \cap \Omega_{a*}) \subseteq \mathsf{D}_0 \cap \Omega_{a*}$ *if and only if* $C \subseteq \mathsf{T}_a$.

Proof. (i) Clear, as any composition in which either all inner functions are constant or the outer function is constant is a constant function.

(ii) Clear, as for any m-ary $g_1, \dots, g_n \in \Omega$ we have $\mathrm{c}_a^{(n)}(g_1, \dots, g_n) = \mathrm{c}_a^{(m)} \in \mathsf{D}_0 \cap \Omega_{a*}$.

(iii) Lemma 7.2, Proposition 7.5(ii), and part (i) imply that $C(\mathsf{D}_0 \cap \Omega_{a*}) \subseteq \mathsf{D}_0 \cap \Omega_{a*}$ whenever $C \subseteq \Omega \cap \mathsf{T}_a = \mathsf{T}_a$.

Assume now that $C \nsubseteq \mathsf{T}_a$. Then there exists a $g \in C$ that does not preserve a, and we have $g(\mathrm{c}_a^{(n)}, \dots, \mathrm{c}_a^{(n)}) = \mathrm{c}_{1-a}^{(n)} \notin \Omega_{a*}$. Therefore $C(\mathsf{D}_0 \cap \Omega_{a*}) \nsubseteq \mathsf{D}_0 \cap \Omega_{a*}$. □

Proposition 7.18. *Let $k \in \mathbb{N}_+$, $a \in \{0,1\}$, and let C be a clone.*

(i) $(\mathsf{D}_k \cap \Omega_{a*})C \subseteq (\mathsf{D}_k \cap \Omega_{a*})$ *if and only if* $C \subseteq \mathsf{L}_0$.
(ii) $C(\mathsf{D}_k \cap \Omega_{a*}) \subseteq (\mathsf{D}_k \cap \Omega_{a*})$ *if and only if* $C \subseteq \mathsf{L}_a$.
(iii) $(\mathsf{D}_k \cap \Omega_{*a})C \subseteq (\mathsf{D}_k \cap \Omega_{*a})$ *if and only if* $C \subseteq \mathsf{L}_1$.
(iv) $C(\mathsf{D}_k \cap \Omega_{*a}) \subseteq (\mathsf{D}_k \cap \Omega_{*a})$ *if and only if* $C \subseteq \mathsf{L}_a$.

Proof. (i) Lemma 7.2 and Propositions 7.5(i) and 7.16(i) imply that $(\mathsf{D}_k \cap \Omega_{a*})C \subseteq \mathsf{D}_k \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_0 = \mathsf{L}_0$. Conversely, if $(\mathsf{D}_k \cap \Omega_{a*})C \subseteq \mathsf{D}_k \cap \Omega_{a*}$, then C includes neither I_1, I^*, Λ_c, V_c, nor SM by Lemma 7.3(g), (i), (j), so $C \subseteq \mathsf{L}_0$.

(ii) Lemma 7.2 and Propositions 7.5(ii) and 7.16(ii) imply that $C(\mathsf{D}_k \cap \Omega_{a*}) \subseteq \mathsf{D}_k \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_a = \mathsf{L}_a$. Conversely, if $C(\mathsf{D}_k \cap \Omega_{a*}) \subseteq \mathsf{D}_k \cap \Omega_{a*}$, then C includes neither $\mathsf{I}_{\overline{a}}$, I^*, Λ_c, V_c, nor SM by Lemma 7.3(a), (d), (e), (f), so $C \subseteq \mathsf{L}_a$.

(iii) Lemma 7.2 and Propositions 7.5(iii) and 7.16(i) imply that $(\mathsf{D}_k \cap \Omega_{*a})C \subseteq \mathsf{D}_k \cap \Omega_{*a}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_1 = \mathsf{L}_1$. Conversely, if $(\mathsf{D}_k \cap \Omega_{*a})C \subseteq \mathsf{D}_k \cap \Omega_{*a}$, then C includes neither I_0, I^*, Λ_c, V_c, nor SM by Lemma 7.3(g), (i), (j), so $C \subseteq \mathsf{L}_1$.

(iv) Lemma 7.2 and Propositions 7.5(iv) and 7.16(ii) imply that $C(\mathsf{D}_k \cap \Omega_{*a}) \subseteq \mathsf{D}_k \cap \Omega_{*a}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_a = \mathsf{L}_a$. Conversely, if $C(\mathsf{D}_k \cap \Omega_{*a}) \subseteq \mathsf{D}_k \cap \Omega_{*a}$, then C includes neither $\mathsf{I}_{\overline{a}}$, I^*, Λ_c, V_c, nor SM by Lemma 7.3(a), (d), (e), (f), so $C \subseteq \mathsf{L}_a$. □

Proposition 7.19. *Let $k \in \mathbb{N}_+$, and let C be a clone.*

(i) *For $k \geq 2$,* $(\mathsf{D}_k \cap \Omega_=)C \subseteq \mathsf{D}_k \cap \Omega_=$ *if and only if* $C \subseteq \mathsf{L}_\mathrm{c}$.
(ii) $(\mathsf{D}_1 \cap \Omega_=)C \subseteq \mathsf{D}_1 \cap \Omega_=$ *if and only if* $C \subseteq \mathsf{LS}$.
(iii) $C(\mathsf{D}_k \cap \Omega_=) \subseteq \mathsf{D}_k \cap \Omega_=$ *if and only if* $C \subseteq \mathsf{L}$.

Proof. (i) Lemma 7.2 and Propositions 7.7(i) and 7.16(i) imply that $(\mathsf{D}_k \cap \Omega_=)C \subseteq \mathsf{D}_k \cap \Omega_=$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_\mathrm{c} = \mathsf{L}_\mathrm{c}$. Conversely, if $(\mathsf{D}_k \cap \Omega_=)C \subseteq \mathsf{D}_k \cap \Omega_=$, then C includes neither I_0, I_1, I^*, Λ_c, V_c, nor SM by Lemma 7.3(i), (j), (l), (m) so $C \subseteq \mathsf{L}_\mathrm{c}$.

(ii) Assume first that $C \subseteq \mathsf{LS}$. Note that $\mathsf{LS} = \mathsf{D}_1 \cap \Omega_{\neq}$ and $\mathsf{L} = \mathsf{D}_1$, and let $f \in (\mathsf{D}_1 \cap \Omega_=)^{(n)}$ and $g_1, \dots, g_n \in (\mathsf{D}_1 \cap \Omega_{\neq})^{(m)}$. The composition $f(g_1, \dots, g_n)$ is a member of L because the outer and inner functions all belong to $\mathsf{D}_1 = \mathsf{L}$. Moreover, it is a sum of an even number of odd polynomials, that is, an even polynomial, so $f(g_1, \dots, g_n) \in \Omega_=$. We conclude that $(\mathsf{D}_1 \cap \Omega_=)C \subseteq \mathsf{D}_1 \cap \Omega_=$.

Assume now that $(\mathsf{D}_1 \cap \Omega_=)C \subseteq \mathsf{D}_1 \cap \Omega_=$. Then C includes neither I_0, I_1, Λ_c, V_c, nor SM by Lemma 7.3(i), (j), (l) so $C \subseteq \mathsf{LS}$.

(iii) Lemma 7.2 and Propositions 7.7(ii) and 7.16(ii) imply that $C(\mathsf{D}_k \cap \Omega_=) \subseteq \mathsf{D}_k \cap \Omega_=$ whenever $C \subseteq \mathsf{L} \cap \Omega = \mathsf{L}$. Conversely, if $C(\mathsf{D}_k \cap \Omega_=) \subseteq \mathsf{D}_k \cap \Omega_=$, then C includes neither Λ_c, V_c, nor SM by Lemma 7.3(e), (f), so $C \subseteq \mathsf{L}$. □

Proposition 7.20. *Let $k \in \mathbb{N}_+$, and let C be a clone.*

(i) *For $k \geq 2$,* $(\mathsf{D}_k \cap \Omega_{\neq})C \subseteq \mathsf{D}_k \cap \Omega_{\neq}$ *if and only if* $C \subseteq \mathsf{L}_\mathrm{c}$.
(ii) $(\mathsf{D}_1 \cap \Omega_{\neq})C \subseteq \mathsf{D}_1 \cap \Omega_{\neq}$ *if and only if* $C \subseteq \mathsf{LS}$.
(iii) $C(\mathsf{D}_k \cap \Omega_{\neq}) \subseteq \mathsf{D}_k \cap \Omega_{\neq}$ *if and only if* $C \subseteq \mathsf{LS}$.

Proof. (i) Lemma 7.2 and Propositions 7.7(iii) and 7.16(i) imply that $(\mathsf{D}_k \cap \Omega_{\neq})C \subseteq \mathsf{D}_k \cap \Omega_{\neq}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_\mathrm{c} = \mathsf{L}_\mathrm{c}$. Conversely, if $(\mathsf{D}_k \cap \Omega_{\neq})C \subseteq \mathsf{D}_k \cap \Omega_{\neq}$, then C includes neither I_0, I_1, I^*, Λ_c, V_c, nor SM by Lemma 7.3(i), (j), (l), (m), so $C \subseteq \mathsf{L}_\mathrm{c}$.

(ii) If $C \subseteq \mathsf{LS}$, then, since $\mathsf{D}_1 \cap \Omega_{\neq} = \mathsf{LS}$ and LS is a clone, it clearly holds that $(\mathsf{D}_1 \cap \Omega_{\neq})C \subseteq \mathsf{LS}\,\mathsf{LS} \subseteq \mathsf{LS} = \mathsf{D}_1 \cap \Omega_{\neq}$. Conversely, if $(\mathsf{D}_1 \cap \Omega_{\neq})C \subseteq \mathsf{D}_1 \cap \Omega_{\neq}$, then C includes neither I_0, I_1, Λ_c, V_c, nor SM by Lemma 7.3(i), (j), (l), so $C \subseteq \mathsf{LS}$.

(iii) Lemma 7.2 and Propositions 7.7(iv) and 7.16(ii) imply that $C(\mathsf{D}_k \cap \Omega_{\neq}) \subseteq \mathsf{D}_k \cap \Omega_{\neq}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{S} = \mathsf{LS}$. Conversely, if $C(\mathsf{D}_k \cap \Omega_{\neq}) \subseteq \mathsf{D}_k \cap \Omega_{\neq}$, then C includes neither I_0, I_1, Λ_{c}, V_{c}, nor SM by Lemma 7.3(b), (e), (f), so $C \subseteq \mathsf{LS}$. $\square$

Proposition 7.21. *Let $k \in \mathbb{N}_+$, $a, b \in \{0,1\}$, and let C be a clone.*

(i) $(\mathsf{D}_k \cap \Omega_{ab})C \subseteq \mathsf{D}_k \cap \Omega_{ab}$ *if and only if* $C \subseteq \mathsf{L}_{\mathrm{c}}$.
(ii) $C(\mathsf{D}_k \cap \Omega_{ab}) \subseteq \mathsf{D}_k \cap \Omega_{ab}$ *if and only if* $C \subseteq \mathsf{L}_a \cap \mathsf{L}_b$.

Proof. (i) Lemma 7.2 and Propositions 7.8(i) and 7.16(i) imply that $(\mathsf{D}_k \cap \Omega_{ab})C \subseteq \mathsf{D}_k \cap \Omega_{ab}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_{\mathrm{c}} = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{D}_k \cap \Omega_{ab})C \subseteq \mathsf{D}_k \cap \Omega_{ab}$, then C includes neither I_0, I_1, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(g), (i), (j), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) Lemma 7.2 and Propositions 7.8(ii) and 7.16(ii) imply that $C(\mathsf{D}_k \cap \Omega_{ab}) \subseteq \mathsf{D}_k \cap \Omega_{ab}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_a \cap \mathsf{T}_b = \mathsf{L}_a \cap \mathsf{L}_b$.

Assume now that $C(\mathsf{D}_k \cap \Omega_{ab}) \subseteq \mathsf{D}_k \cap \Omega_{ab}$. Then C includes neither I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(d), (e), (f). If $a = b$, then C does not include $\mathsf{I}_{\overline{a}}$ by Lemma 7.3(a), so $C \subseteq \mathsf{L}_a = \mathsf{L}_a \cap \mathsf{L}_b$. If $a \neq b$, then C includes neither I_0 nor I_1 by Lemma 7.3(c), so $C \subseteq \mathsf{L}_{\mathrm{c}} = \mathsf{L}_a \cap \mathsf{L}_b$. $\square$

Proposition 7.22. *Let $i, j \in \mathbb{N}_+$ with $i > j \geq 1$, and let C be a clone.*

(i) $(\mathsf{D}_i \cap \mathsf{X}_j)C \subseteq \mathsf{D}_i \cap \mathsf{X}_j$ *if and only if* $C \subseteq \mathsf{LS}$.
(ii) $C(\mathsf{D}_i \cap \mathsf{X}_j) \subseteq \mathsf{D}_i \cap \mathsf{X}_j$ *if and only if* $C \subseteq \mathsf{L}$.

Proof. (i) Lemma 7.2 and Propositions 7.10(i), (ii) and 7.16(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_j)C \subseteq \mathsf{D}_i \cap \mathsf{X}_j$ whenever $C \subseteq \mathsf{LS} \cap \mathsf{L} = \mathsf{LS}$ if $k \geq 2$ and whenever $C \subseteq \mathsf{S} \cap \mathsf{L} = \mathsf{LS}$ if $k = 1$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_j)C \subseteq \mathsf{D}_i \cap \mathsf{X}_j$, then C includes neither I_0, I_1, Λ_{c}, V_{c}, nor SM by Lemma 7.3(h), (i), (j), so $C \subseteq \mathsf{LS}$.

(ii) Lemma 7.2 and Propositions 7.10(iii) and 7.16(ii) imply that $C(\mathsf{D}_i \cap \mathsf{X}_j) \subseteq \mathsf{D}_i \cap \mathsf{X}_j$ whenever $C \subseteq \mathsf{L} \cap \mathsf{L} = \mathsf{L}$. Conversely, if $C(\mathsf{D}_i \cap \mathsf{X}_j) \subseteq \mathsf{D}_i \cap \mathsf{X}_j$, then C includes neither Λ_{c}, V_{c}, nor SM by Lemma 7.3(e), (f), so $C \subseteq \mathsf{L}$. $\square$

Proposition 7.23. *Let $i, j \in \mathbb{N}_+$ with $i > j \geq 1$, $a \in \{0,1\}$, and let C be a clone.*

(i) $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$ *if and only if* $C \subseteq \mathsf{L}_{\mathrm{c}}$.
(ii) $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$ *if and only if* $C \subseteq \mathsf{L}_a$.
(iii) $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$ *if and only if* $C \subseteq \mathsf{L}_{\mathrm{c}}$.
(iv) $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$ *if and only if* $C \subseteq \mathsf{L}_a$.

Proof. (i) Lemma 7.2 and Propositions 7.5(i) and 7.22(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{LS} \cap \mathsf{T}_0 = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$, then C includes neither I_0, I_1, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(g), (h), (i), (j), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) Lemma 7.2 and Propositions 7.5(ii) and 7.22(ii) imply that $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_a = \mathsf{L}_a$. Conversely, if $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$, then C includes neither $\mathsf{I}_{\overline{a}}$, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(a), (d), (e), (f), so $C \subseteq \mathsf{L}_a$.

(iii) Lemma 7.2 and Propositions 7.5(iii) and 7.22(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$ whenever $C \subseteq \mathsf{LS} \cap \mathsf{T}_1 = \mathsf{L}_{\mathrm{c}}$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$, then C includes neither I_0, I_1, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(g), (h), (i), (j), so $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(iv) Lemma 7.2 and Propositions 7.5(iv) and 7.22(ii) imply that $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{T}_a = \mathsf{L}_a$. Conversely, if $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$, then C includes neither $\mathsf{I}_{\overline{a}}$, I^*, Λ_{c}, V_{c}, nor SM by Lemma 7.3(a), (d), (e), (f), so $C \subseteq \mathsf{L}_a$. $\square$

Proposition 7.24. *Let $i, j \in \mathbb{N}_+$ with $i > j \geq 1$, and let C be a clone.*

(i) *For $j \geq 2$,* $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}$ *if and only if* $C \subseteq \mathsf{L}_{\mathrm{c}}$.

(ii) $(\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_=)C \subseteq \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_=$ *if and only if* $C \subseteq \mathsf{LS}$.
(iii) $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=$ *if and only if* $C \subseteq \mathsf{L}$.

Proof. (i) Lemma 7.2 and Propositions 7.7(ii) and 7.22(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=)C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=$ whenever $C \subseteq \mathsf{LS} \cap \mathsf{T_c} = \mathsf{L_c}$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=)C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=$, then C includes neither I_0, I_1, I^*, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(i), (j), (l), (m), so $C \subseteq \mathsf{L_c}$.

(ii) Lemma 7.2 and Propositions 7.13(ii) and 7.16(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_=)C \subseteq \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_=$ whenever $C \subseteq \mathsf{S} \cap \mathsf{L} = \mathsf{LS}$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_=)C \subseteq \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_=$, then C includes neither I_0, I_1, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(i), (j), (l), so $C \subseteq \mathsf{LS}$.

(iii) Lemma 7.2 and Propositions 7.7(ii) and 7.22(ii) imply that $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=$ whenever $C \subseteq \mathsf{L} \cap \Omega = \mathsf{L}$. Conversely, if $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_=$, then C includes neither Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(e), (f), so $C \subseteq \mathsf{L}$. ☐

Proposition 7.25. *Let $i, j \in \mathbb{N}_+$ with $i > j \geq 1$, and let C be a clone.*

(i) *For $j \geq 2$,* $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$ *if and only if* $C \subseteq \mathsf{L_c}$.
(ii) $(\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\neq})C \subseteq \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\neq}$ *if and only if* $C \subseteq \mathsf{LS}$.
(iii) $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$ *if and only if* $C \subseteq \mathsf{LS}$.

Proof. (i) Lemma 7.2 and Propositions 7.7(iii) and 7.22(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$ whenever $C \subseteq \mathsf{LS} \cap \mathsf{T_c} = \mathsf{L_c}$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$, then C includes neither I_0, I_1, I^*, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(i), (j), (l), (m), so $C \subseteq \mathsf{L_c}$.

(ii) Lemma 7.2 and Propositions 7.14(ii) and 7.16(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\neq})C \subseteq \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\neq}$ whenever $C \subseteq \mathsf{S} \cap \mathsf{L} = \mathsf{LS}$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\neq})C \subseteq \mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\neq}$, then C includes neither I_0, I_1, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(i), (j), (l), so $C \subseteq \mathsf{LS}$.

(iii) Lemma 7.2 and Propositions 7.7(iv) and 7.22(ii) imply that $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$ whenever $C \subseteq \mathsf{L} \cap \mathsf{S} = \mathsf{LS}$. Conversely, if $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\neq}$, then C includes neither I_0, I_1, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(b), (e), (f), so $C \subseteq \mathsf{LS}$. ☐

Proposition 7.26. *Let $i, j \in \mathbb{N}_+$ with $i > j \geq 1$, $a, b \in \{0, 1\}$, and let C be a clone.*

(i) $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$ *if and only if* $C \subseteq \mathsf{L_c}$.
(ii) $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$ *if and only if* $C \subseteq \mathsf{L}_a \cap \mathsf{L}_b$.

Proof. (i) Lemma 7.2 and Propositions 7.10(i), (ii) and 7.21(i) imply that $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$ whenever $C \subseteq \mathsf{L_c} \cap \mathsf{LS} = \mathsf{L_c}$ if $j \geq 2$ and whenever $C \subseteq \mathsf{L_c} \cap \mathsf{S} = \mathsf{L_c}$ if $j = 1$. Conversely, if $(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab})C \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$, then C includes neither I_0, I_1, I^*, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(g), (i), (j), so $C \subseteq \mathsf{L_c}$.

(ii) Lemma 7.2 and Propositions 7.10(iii) and 7.21(ii) imply that $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$ whenever $C \subseteq \mathsf{L}_a \cap \mathsf{L}_b \cap \mathsf{L} = \mathsf{L}_a \cap \mathsf{L}_b$.

Assume now that $C(\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}) \subseteq \mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$. If $a = b$, then C includes neither $\mathsf{I}_{\overline{a}}$, I^*, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(a), (d), (e), (f), so $C \subseteq \mathsf{L}_a = \mathsf{L}_a \cap \mathsf{L}_b$. If $a \neq b$, then C includes neither I_0, I_1, I^*, Λ_c, $\mathsf{V_c}$, nor SM by Lemma 7.3(c), (d), (e), (f), so $C \subseteq \mathsf{L_c} = \mathsf{L}_a \cap \mathsf{L}_b$. ☐

Proof of Theorem 7.1. The theorem puts together Propositions 7.4, 7.5, 7.7, 7.8, 7.10, 7.11, 7.12, 7.13, 7.14, 7.15, 7.16, 7.17, 7.18, 7.19, 7.20, 7.21, 7.22, 7.23, 7.24, 7.25, 7.26. ☐

With the help of Post's lattice (Figure 1) and by reading off from Table 2, we can determine for any pair (C_1, C_2) of clones which $\mathsf{L_c}$-stable classes are (C_1, C_2)-stable. If $\mathsf{L_c} \subseteq C_2$, then any (C_1, C_2)-stable class is $(\mathsf{I_c}, \mathsf{L_c})$-stable by Lemma 2.16 and hence also $\mathsf{L_c}$-stable by Lemma 6.2. Therefore, in the case when $\mathsf{L_c} \subseteq C_2$, the (C_1, C_2)-stable classes are among the $\mathsf{L_c}$-stable ones and they can be easily picked out from Table 2. In particular, we have an explicit description of $(\mathsf{I}_{\mathrm{c}}, C)$-stable classes ("clonoids" of Aichinger and Mayr [1]) and C-stable classes for $\mathsf{L}_{\mathrm{c}} \subseteq C$. The L_0-stable classes (see Corollary 7.28(iii)) were determined earlier by Kreinecker [10, Theorem 3.12].

Corollary 7.27.

(i) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{L}_{\mathrm{c}})$*-stable classes are* Ω, Ω_{a*}, Ω_{*a}, $\Omega_{\approx}$, Ω_{ab}, D_k, $\mathsf{D}_k \cap \Omega_{a*}$, $\mathsf{D}_k \cap \Omega_{*a}$, $\mathsf{D}_k \cap \Omega_{\approx}$, $\mathsf{D}_k \cap \Omega_{ab}$, X_k, $\mathsf{X}_k \cap \Omega_{a*}$, $\mathsf{X}_k \cap \Omega_{*a}$, $\mathsf{X}_k \cap \Omega_{\approx}$, $\mathsf{X}_k \cap \Omega_{ab}$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$, D_0, $\mathsf{D}_0 \cap \Omega_{a*}$, $\emptyset$, *for* $a, b \in \{0, 1\}$, $\approx \in \{=, \neq\}$, *and* $i, j, k \in \mathbb{N}_+$ *with* $i > j \geq 1$.

(ii) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{LS})$*-stable classes are* Ω, $\Omega_{\approx}$, X_k, $\mathsf{X}_k \cap \Omega_{\approx}$, D_k, $\mathsf{D}_k \cap \Omega_{\approx}$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}$, D_0, $\emptyset$, *for* $\approx \in \{=, \neq\}$, *and* $i, j, k \in \mathbb{N}_+$ *with* $i > j \geq 1$.

(iii) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{L}_0)$*-stable classes are* Ω, Ω_{0*}, Ω_{*0}, $\Omega_{=}$, Ω_{00}, X_k, $\mathsf{X}_k \cap \Omega_{0*}$, $\mathsf{X}_k \cap \Omega_{*0}$, $\mathsf{X}_k \cap \Omega_{=}$, $\mathsf{X}_k \cap \Omega_{00}$, D_k, $\mathsf{D}_k \cap \Omega_{0*}$, $\mathsf{D}_k \cap \Omega_{*0}$, $\mathsf{D}_k \cap \Omega_{=}$, $\mathsf{D}_k \cap \Omega_{00}$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{0*}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*0}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{00}$, D_0, $\mathsf{D}_0 \cap \Omega_{0*}$, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(iv) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{L}_1)$*-stable classes are* Ω, Ω_{1*}, Ω_{*1}, $\Omega_{=}$, Ω_{11}, X_k, $\mathsf{X}_k \cap \Omega_{1*}$, $\mathsf{X}_k \cap \Omega_{*1}$, $\mathsf{X}_k \cap \Omega_{=}$, $\mathsf{X}_k \cap \Omega_{11}$, D_k, $\mathsf{D}_k \cap \Omega_{1*}$, $\mathsf{D}_k \cap \Omega_{*1}$, $\mathsf{D}_k \cap \Omega_{=}$, $\mathsf{D}_k \cap \Omega_{11}$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{1*}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*1}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{11}$, D_0, $\mathsf{D}_0 \cap \Omega_{1*}$, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(v) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{L})$*-stable classes are* Ω, $\Omega_{=}$, X_k, $\mathsf{X}_k \cap \Omega_{=}$, D_k, $\mathsf{D}_k \cap \Omega_{=}$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{=}$, D_0, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(vi) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{S}_{\mathrm{c}})$*-stable classes are* Ω, Ω_{a*}, Ω_{*a}, $\Omega_{\approx}$, Ω_{ab}, $\mathsf{X}_1 \cap \Omega_{\approx}$, $\mathsf{X}_1 \cap \Omega_{ab}$, D_0, $\mathsf{D}_0 \cap \Omega_{a*}$, $\emptyset$, *for* $a, b \in \{0, 1\}$ *and* $\approx \in \{=, \neq\}$.

(vii) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{S})$*-stable classes are* Ω, $\Omega_{\approx}$, $\mathsf{X}_1 \cap \Omega_{\approx}$, D_0, $\emptyset$, *for* $\approx \in \{=, \neq\}$.

(viii) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{T}_{\mathrm{c}})$*-stable classes are* Ω, Ω_{a*}, Ω_{*a}, $\Omega_{=}$, Ω_{ab}, $\mathsf{X}_1 \cap \Omega_{=}$, $\mathsf{X}_1 \cap \Omega_{aa}$, D_0, $\mathsf{D}_0 \cap \Omega_{a*}$, $\emptyset$, *for* $a, b \in \{0, 1\}$

(ix) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{T}_0)$*-stable classes are* Ω, Ω_{0*}, Ω_{*0}, $\Omega_{=}$, Ω_{00}, $\mathsf{X}_1 \cap \Omega_{=}$, $\mathsf{X}_1 \cap \Omega_{00}$, D_0, $\mathsf{D}_0 \cap \Omega_{0*}$, $\emptyset$.

(x) *The* $(\mathsf{I}_{\mathrm{c}}, \mathsf{T}_1)$*-stable classes are* Ω, Ω_{1*}, Ω_{*1}, $\Omega_{=}$, Ω_{11}, $\mathsf{X}_1 \cap \Omega_{=}$, $\mathsf{X}_1 \cap \Omega_{11}$, D_0, $\mathsf{D}_0 \cap \Omega_{1*}$, $\emptyset$.

(xi) *The* $(\mathsf{I}_{\mathrm{c}}, \Omega)$*-stable classes are* Ω, $\Omega_{=}$, $\mathsf{X}_1 \cap \Omega_{=}$, D_0, $\emptyset$.

Corollary 7.28.

(i) *The* L_{c}*-stable classes are* Ω, Ω_{a*}, Ω_{*a}, $\Omega_{\approx}$, Ω_{ab}, D_k, $\mathsf{D}_k \cap \Omega_{a*}$, $\mathsf{D}_k \cap \Omega_{*a}$, $\mathsf{D}_k \cap \Omega_{\approx}$, $\mathsf{D}_k \cap \Omega_{ab}$, X_k, $\mathsf{X}_k \cap \Omega_{a*}$, $\mathsf{X}_k \cap \Omega_{*a}$, $\mathsf{X}_k \cap \Omega_{\approx}$, $\mathsf{X}_k \cap \Omega_{ab}$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{a*}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{*a}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{\approx}$, $\mathsf{D}_i \cap \mathsf{X}_j \cap \Omega_{ab}$, D_0, $\mathsf{D}_0 \cap \Omega_{a*}$, $\emptyset$, *for* $a, b \in \{0, 1\}$, $\approx \in \{=, \neq\}$, *and* $i, j, k \in \mathbb{N}_+$ *with* $i > j \geq 1$.

(ii) *The* LS*-stable classes are* Ω, X_k, $\mathsf{X}_1 \cap \Omega_{\approx}$, D_k, $\mathsf{D}_1 \cap \Omega_{\approx}$, $\mathsf{D}_i \cap \mathsf{X}_j$, $\mathsf{D}_i \cap \mathsf{X}_1 \cap \Omega_{\approx}$, D_0, $\emptyset$, *for* $\approx \in \{=, \neq\}$ *and* $i, j, k \in \mathbb{N}_+$ *with* $i > j \geq 1$.

(iii) *The* L_0*-stable classes are* Ω, Ω_{0*}, D_k, $\mathsf{D}_k \cap \Omega_{0*}$, D_0, $\mathsf{D}_0 \cap \Omega_{0*}$, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(iv) *The* L_1*-stable classes are* Ω, Ω_{*1}, D_k, $\mathsf{D}_k \cap \Omega_{*1}$, D_0, $\mathsf{D}_0 \cap \Omega_{1*}$, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(v) *The* L*-stable classes are* Ω, D_k, D_0, $\emptyset$, *for* $k \in \mathbb{N}_+$.

(vi) *The* S_{c}*-stable classes are* Ω, Ω_{a*}, Ω_{*a}, $\Omega_{\approx}$, Ω_{ab}, $\mathsf{X}_1 \cap \Omega_{\approx}$, $\mathsf{X}_1 \cap \Omega_{ab}$, D_0, $\mathsf{D}_0 \cap \Omega_{a*}$, $\emptyset$, *for* $a, b \in \{0, 1\}$ *and* $\approx \in \{=, \neq\}$.

(vii) *The* S*-stable classes are* Ω, $\mathsf{X}_1 \cap \Omega_{\approx}$, D_0, $\emptyset$, *for* $\approx \in \{=, \neq\}$.

(viii) *The* T_{c}*-stable classes are* Ω, Ω_{a*}, Ω_{*a}, $\Omega_{=}$, Ω_{ab}, D_0, $\mathsf{D}_0 \cap \Omega_{a*}$, $\emptyset$, *for* $a, b \in \{0, 1\}$

(ix) *The* T_0*-stable classes are* Ω, Ω_{0*}, D_0, $\mathsf{D}_0 \cap \Omega_{0*}$, $\emptyset$.

(x) *The* T_1*-stable classes are* Ω, Ω_{*1}, D_0, $\mathsf{D}_0 \cap \Omega_{1*}$, $\emptyset$.

(xi) *The* Ω*-stable classes are* Ω, D_0, $\emptyset$.

Recall from Lemma 6.2(iii) that $(\mathsf{I}_{\mathrm{c}}, \mathsf{L}_{\mathrm{c}})$-stability is equivalent to L_{c}-stability. Therefore, as expected, the classes listed in Corollary 7.27(i) are the same as those in Corollary 7.28(i). By comparing Corollary 7.27(vi) with Corollary 7.28(vi), we see also that $(\mathsf{I}_c, \mathsf{S}_c)$-stability is equivalent to S_c-stability. Whether the reason for this is a relationship similar to Lemma 6.2 is beyond the scope of this paper.

Corollary 7.29. *S_c-stability is equivalent to $(\mathsf{I}_c, \mathsf{S}_c)$-stability.*

8. Final remarks and perspectives

Looking into directions of future research, one may consider arbitrary pairs of clones C_1 and C_2 on arbitrary sets A and B and describe the (C_1, C_2)-stable sets in this case. However, this task is challenging. Firstly, there are uncountably many clones on sets with at least three elements (see [20]), and not all of them are known. Secondly, for given clones C_1 and C_2, there may be uncountably many (C_1, C_2)-stable classes, in which case an explicit description may be unattainable.

For this reason, a natural next step would be to consider (C_1, C_2)-stability for clones C_1 and C_2 on the two-element set $\{0, 1\}$, which are well known (see Post [15]). Moreover, the cardinality of the closure system of (I_c, C)-stable classes of Boolean functions is known for every clone C on $\{0, 1\}$, due to the following result by Sparks [19]. However, this result does not provide an explicit description of the (I_c, C)-stable classes, even for the cases where the number of (I_c, C)-stable classes is finite.

Theorem 8.1 ([19, Theorem 1.3])**.** *Let A be a finite set with $|A| > 1$, and let $B := \{0, 1\}$. Denote by J_A the clone of projections on A, and let C be a clone on B. Then the following statements hold.*

(i) *$\mathcal{L}_{(\mathsf{J}_A, C)}$ is finite if and only if C contains a near-unanimity operation.*
(ii) *$\mathcal{L}_{(\mathsf{J}_A, C)}$ is countably infinite if and only if C contains a Mal'cev operation but no majority operation.*
(iii) *$\mathcal{L}_{(\mathsf{J}_A, C)}$ has the cardinality of the continuum if and only if C contains neither a near-unanimity operation nor a Mal'cev operation.*

Recall that an n-ary operation $f \in \mathcal{O}_B$ with $n \geq 3$ is called a *near-unanimity operation* if $f(x, \ldots, x, y, x, \ldots, x) = x$ for all $x, y \in B$, where the single occurrence of y can occur in any of the n argument positions. A ternary near-unanimity operation is called a *majority operation.* A ternary operation $f \in \mathcal{O}_B$ is called a *Mal'cev operation* if $f(y, y, x) = f(x, y, y) = x$ for all $x, y \in B$.

A clone C on $\{0, 1\}$ contains a Mal'cev operation but no majority operation (statement (ii)) if and only if $\mathsf{L}_c \subseteq C \subseteq \mathsf{L}$; this situation is completely described in the current paper. In view of Theorem 8.1, explicit descriptions of the (C_1, C_2)-stable classes of Boolean functions seem attainable in the case when C_2 contains a near-unanimity function (statement (i)), but this may not be the case when C_2 contains neither a near-unanimity operation nor a Mal'cev operation (statement (iii)). This suggests a feasible direction for future research.

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